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# A CONTINUUM MODEL FOR CURVILINEAR LAMINATED COMPOSITES†

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**Abstract**—The governing kinematical, dynamical and constitutive equations of an approximate theory for a curvilinear laminated body are presented. Starting with two-term expansions of the field variables about the midsurfaces of the discrete layers, it is shown that in the first approximation the deformation of a curvilinear laminated body is described by three vector fields, termed the gross motion and the local deformations. Dynamical balance laws are derived for the resultant stresses and moments of stress. A constitutive theory is formulated for nonlinear elastic materials. A simplified version of linear constitutive equations is discussed and specific forms of the balance laws and constitutive equations are given for cylindrical and spherical laminated bodies.

## NOTATION

Throughout this paper we use standard vector and tensor notation. Bold face letters indicate vector or tensor quantities. Upper case italic subscripts assume the values 1, 2, 3 and indicate tensors in the Lagrangian system $X^K$. They are raised and lowered by the metric tensor $G_{KL}$ and its inverse $G^{KL}$. Greek subscripts and superscripts assume the values 1, 3 and are referred to the $X^1$, $X^3$ components of the Lagrangian system. Lower case italic subscripts assume the values 1, 2, 3 and specify tensors in the Eulerian system $x^l$. They are raised and lowered by the metric tensor $g_{kl}$ and its inverse $g^{kl}$. Superscripts and subscripts in parentheses indicate whether a quantity belongs to a reinforcing layer or a matrix layer; they are not tensor indices.

## 1. INTRODUCTION

In a recent series of publications [1–4] an approximate theory for a laminated body, fabricated by alternating a matrix layer and a fiber-reinforcing layer, has been derived and analyzed. The theories discussed in Refs. [1–4] have been restricted to plane laminae. In this paper we generalize the theory of a laminated body to include layering whose geometric structure is not necessarily planar. We consider a laminated solid formed by the compounding of matrix layers and fiber-reinforcing layers in such a way that the interfaces of the layers are parallel surfaces. As examples of such bodies we mention laminated cylinders of arbitrary cross section, laminated spherical shells and laminated domes.

Conceptually, the states of deformation and stress in laminated bodies can be determined by solving the governing system of balance laws within each layer, and by satisfying various continuity conditions at the interfaces and certain boundary conditions on the bounding surfaces. In practice this is impossible to carry out. Fortunately, in many applications one is not interested in the detailed deformation within each layer but only in certain gross quantities. In that case the system of discrete layers can be approximated by a homogeneous

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continuum model. This is accomplished by representing the motion within each layer by a two-term expansion and by averaging over the thicknesses of the layers. In this continuum model of a laminated body the kinematics is described by three vector fields: the gross motion and two local or "micro" motions. By requiring that the motion is continuous at the interfaces of the layering it is shown that these kinematical variables satisfy a constraint condition. In Section 2 we present the geometrical and kinematical ideas necessary for the development of a continuum model of a curvilinear laminated body. In Section 3 we derive the corresponding balance laws by integrating Cauchy's balance laws of linear momentum and moments of these equations and passing to the continuum model. This process is justifiable if the characteristic length of the deformation is large compared to the thicknesses of the layers and if the ratio of the thicknesses of the layers to the minimal principal radius of curvature of the lamina is small compared to unity.

In Section 4 we formulate a constitutive theory for nonlinear elastic material with a lamellar structuring.† In Section 5 the equations are linearized for isotropic materials. In Section 6 we present the constraint conditions, balance laws and linear constitutive equations for cylindrical laminated bodies and for spherical laminated bodies.

## 2. GEOMETRY AND DEFORMATION

We consider a system of parallel surfaces and choose a coordinate system $X^K$ of the undeformed body, such that $X^2$ is perpendicular to the family of surfaces. The coordinates $X^1, X^3$ are taken as surface coordinates of one of the surfaces (the surface $X^2 = 0$) whose unit normal vector is $\mathbf{v}$. In a cartesian system $\mathbf{Z}$ the family of surfaces is defined by

$$\mathbf{Z} = \hat{\mathbf{Z}}(X^1, X^3) + X^2\mathbf{v}(X^1, X^3). \tag{2.1}$$

Since $\mathbf{v}$ is a normal to the surface $X^2 = 0$, we have

$$\partial_\alpha\hat{\mathbf{Z}}\cdot\mathbf{v} = 0, \qquad \alpha = 1, 3. \tag{2.2}$$

We introduce the base vectors $\mathbf{G}_K$ of the coordinate system $X^K$ as

$$\begin{aligned} \mathbf{G}_\alpha &= \partial_\alpha\mathbf{Z} \\ \mathbf{G}_2 &= \partial_2 Z_1 = \mathbf{v}. \end{aligned} \tag{2.3}$$

From (2.1) and (2.2) it follows that the metric tensor $G_{KL} = \mathbf{G}_K\cdot\mathbf{G}_L$ has the form:

$$\begin{aligned} G_{\alpha\beta} &= \mathbf{G}_\alpha\cdot\mathbf{G}_\beta = \hat{G}_{\alpha\beta} - 2X^2\hat{B}_{\alpha\beta} + (X^2)^2\hat{B}^\gamma_\alpha\hat{B}_{\gamma\beta} \\ G_{2\beta} &= G_{\beta 2} = 0 \\ G_{22} &= 1, \end{aligned} \tag{2.4}$$

where $\hat{G}_{\alpha\beta}$ is the metric tensor of the surface $X^2 = 0$, and $\hat{B}_{\alpha\beta}$ is its second fundamental form:

$$\hat{G}_{\alpha\beta} = \hat{\mathbf{G}}_\alpha\cdot\hat{\mathbf{G}}_\beta, \qquad \hat{\mathbf{G}}_\alpha = \partial_\alpha\hat{\mathbf{Z}}, \quad \hat{G}_{\alpha\gamma}\hat{G}^{\gamma\beta} = \delta^\gamma_\alpha \tag{2.5}$$

$$\hat{B}_{\alpha\beta} = \partial_\beta\hat{\mathbf{G}}_\alpha\cdot\mathbf{v} = -\hat{\mathbf{G}}_\alpha\cdot\partial_\beta\mathbf{v}, \qquad \hat{B}_\alpha{}^\beta = \hat{B}_{\alpha\gamma}\hat{G}^{\gamma\beta}. \tag{2.6}$$

† The theory formulated here resembles a micromorphic material with internal constraints.

If we define

$$G \equiv \det(G_{\alpha\beta})$$
$$\hat{G} \equiv \det(\hat{G}_{\alpha\beta}), \tag{2.7}$$

then it can be shown (see Thomas [5, p. 105]), that

$$\sqrt{G} = \sqrt{(\hat{G})}K^*, \tag{2.8}$$

where

$$K^* = 1 - 2X^2\hat{B}^\gamma{}_\gamma + (X^2)^2\frac{\hat{B}}{\hat{G}} \tag{2.9}$$

and

$$\hat{B} = \det \hat{B}_{\alpha\beta}. \tag{2.10}$$

In the sequel we will need the following relations for the Christoffel symbols of the second kind:

$$\left\{\begin{matrix} K \\ 2 \quad 2 \end{matrix}\right\}_{\mathbf{G}} = 0 \qquad \left\{\begin{matrix} 2 \\ 2 \quad K \end{matrix}\right\}_{\mathbf{G}} = 0$$
$$K^*\left\{\begin{matrix} K \\ K \quad \alpha \end{matrix}\right\}_{\mathbf{G}} = K^*\left\{\begin{matrix} \beta \\ \beta \quad \alpha \end{matrix}\right\}_{\hat{\mathbf{G}}} + \partial_\alpha K^* \tag{2.11}$$
$$K^*\left\{\begin{matrix} K \\ K \quad 2 \end{matrix}\right\}_{\mathbf{G}} = \partial_2 K^*.$$

We consider a laminated composite which before deformation consists of alternating parallel sheets of a matrix and fiber-reinforcing layers. We choose the curvilinear coordinates $X^K$ such that the coordinates $X^1$, $X^3$ are surface coordinates of a family of parallel surfaces, discrete members of which coincide with the interfaces of the lamination, and the coordinate $X^2$ is in the direction of the normal to the lamination. The system $X^K$ is defined by equations (2.1) and (2.3). We assume that the thickness of the fiber is $d_f$ and the thickness of the matrix is $d_m$. We fix our attention on the $k$th pair of layers and introduce the local coordinates $\bar{X}^2_{(f)}$ and $\bar{X}^2_{(m)}$ at the center of the $k$th fiber and $k$th matrix (see Fig. 1).

Within the $k$th pair of layers we express (2.1) in terms of the relative coordinate system

$$\mathbf{Z}_{(fk)} = \bar{\mathbf{Z}}_{(fk)}(X^1, X^2_{(fk)}, X^3) + \bar{X}^2_{(f)}\mathbf{v}(X^1, X^3)$$
$$\mathbf{Z}_{(mk)} = \bar{\mathbf{Z}}_{(mk)}(X^1, X^2_{(mk)}, X^3) + \bar{X}^2_{(m)}\mathbf{v}(X^1, X^3), \tag{2.12}$$

where $\bar{\mathbf{Z}}_{(fk)}$ and $\bar{\mathbf{Z}}_{(mk)}$ satisfy

$$\bar{\mathbf{Z}}_{(fk)} - \bar{\mathbf{Z}}_{(mk)} = \tfrac{1}{2}(d_f + d_m)\mathbf{v}. \tag{2.13}$$

We define the basic vectors within the $k$th reinforcing and matrix layers as

$$\mathbf{G}_{(fk)\alpha} = \partial_\alpha \mathbf{Z}_{(fk)}, \qquad \mathbf{G}_{(fk)2} = \mathbf{v}$$
$$\mathbf{G}_{(mk)\alpha} = \partial_\alpha \mathbf{Z}_{(mk)}, \qquad \mathbf{G}_{(mk)2} = \mathbf{v}, \tag{2.14}$$

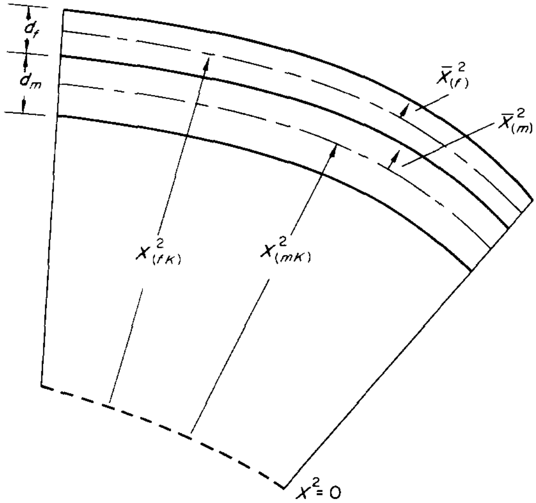


FIG. 1. Element of the $k$th pair of layers.

where $\alpha = 1, 3$, and also the basic vectors at the corresponding midsurfaces

$$\begin{aligned} \bar{\mathbf{G}}_{(fk)\alpha} &= \partial_\alpha \bar{\mathbf{Z}}_{(fk)}, & \bar{\mathbf{G}}_{(fk)2} &= \mathbf{v} \\ \bar{\mathbf{G}}_{(mk)\alpha} &= \partial_\alpha \bar{\mathbf{Z}}_{(mk)}, & \bar{\mathbf{G}}_{(mk)2} &= \mathbf{v}. \end{aligned} \tag{2.15}$$

If we define $G_{(fk)}, G_{(mk)}$

$$\begin{aligned} G_{(fk)} &= \det(G_{(fk)\alpha\beta}); & G_{(fk)\alpha\beta} &= \mathbf{G}_{(fk)\alpha} \cdot \mathbf{G}_{(fk)\beta} \\ G_{(mk)} &= \det(G_{(mk)\alpha\beta}); & G_{(mk)\alpha\beta} &= \mathbf{G}_{(mk)\alpha} \cdot \mathbf{G}_{(mk)\beta} \end{aligned} \tag{2.16}$$

and $\bar{G}_{(fk)}, \bar{G}_{(mk)}$

$$\begin{aligned} \bar{G}_{(fk)} &= \det(\bar{G}_{(fk)\alpha\beta}); & \bar{G}_{(fk)\alpha\beta} &= \bar{\mathbf{G}}_{(fk)\alpha} \cdot \bar{\mathbf{G}}_{(fk)\beta} \\ \bar{G}_{(mk)} &= \det(\bar{G}_{(mk)\alpha\beta}); & \bar{G}_{(mk)\alpha\beta} &= \bar{\mathbf{G}}_{(mk)\alpha} \cdot \bar{\mathbf{G}}_{(mk)\beta}, \end{aligned} \tag{2.17}$$

we obtain from (2.8) that

$$\begin{aligned} \sqrt{(G_{(fk)})} &= \sqrt{(\bar{G}_{(fk)})} K^*_{(fk)} \\ \sqrt{(G_{(mk)})} &= \sqrt{(\bar{G}_{(mk)})} K^*_{(mk)}, \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} K^*_{(fk)} &= 1 - 2\bar{X}^2_{(f)} \bar{B}_{(fk)}{}^\gamma{}_\gamma + (X^2_{(f)})^2 \frac{\bar{B}_{(fk)}}{\bar{G}_{(fk)}} \\ K^*_{(mk)} &= 1 - 2\bar{X}^2_{(m)} \bar{B}_{(mk)}{}^\gamma{}_\gamma + (X^2_{(m)})^2 \frac{\bar{B}_{(mk)}}{\bar{G}_{(mk)}}. \end{aligned} \tag{2.19}$$

In equations (2.19), we have defined

$$\begin{aligned} \bar{B}_{(fk)\alpha\beta} &= \partial_\beta \bar{\mathbf{G}}_{(fk)\alpha} \cdot \mathbf{v}; & \bar{B}_{(fk)} &= \det(\bar{B}_{(fk)\alpha\beta}) \\ \bar{B}_{(mk)\alpha\beta} &= \partial_\beta \bar{\mathbf{G}}_{(mk)\alpha} \cdot \mathbf{v}; & \bar{B}_{(mk)} &= \det(\bar{B}_{(mk)\alpha\beta}). \end{aligned} \tag{2.20}$$

If the characteristic length of the deformation is large compared to the thicknesses of the layers, then within the $k$th pair of layers we can approximate the motion by

$$\begin{aligned} x^l_{(fk)} &\simeq \bar{x}^l_{(fk)}(X^1, X^2_{(fk)}, X^3, t) + \bar{X}^2_{(f)}\psi_{(fk)2}{}^l(X^1, X^2_{(fk)}, X^3, t) \\ x^l_{(mk)} &\simeq \bar{x}^l_{(mk)}(X^1, X^2_{(mk)}, X^3, t) + \bar{X}^2_{(m)}\psi_{(mk)2}{}^l(X^1, X^2_{(mk)}, X^3, t), \end{aligned} \tag{2.21}$$

where we have referred the motion to an arbitrary coordinate system $(x^l)$ of the deformed body. In equations (2.21), $\bar{x}^l_{(fk)}$ and $\bar{x}^l_{(mk)}$ represent the motion of the midsurface of the reinforcing and matrix layers, respectively; $\psi_{(fk)2}{}^l$ and $\psi_{(mk)2}{}^l$ give the antisymmetric shear and symmetric thickness stretch deformations of the fiber and matrix, respectively.

If we change the coordinate system $(x^l)$ of the deformed body to a coordinate system $(y^l)$ where the transformation is given by

$$y^j = h^j(x^l), \tag{2.22}$$

then in the system $(y^i)$ we can write the approximate motion (2.21) as

$$\begin{aligned} y^l_{(fk)} &\simeq \bar{y}^l_{(fk)}(X^1, X^2_{(fk)}, X^3, t) + \bar{X}^2_{(f)}\varphi_{2(fk)}{}^l(X^1, X^2_{(fk)}, X^3, t) \\ y^l_{(mk)} &\simeq \bar{y}^l_{(mk)}(X^1, X^2_{(mk)}, X^3, t) + \bar{X}^2_{(m)}\varphi_{2(mk)}{}^l(X^1, X^2_{(mk)}, X^3, t). \end{aligned} \tag{2.23}$$

Using (2.21) and (2.22), we obtain

$$\begin{aligned} y^l_{(fk)} &\simeq h^l(\bar{x}^j_{(fk)} + \bar{X}^2_{(f)}\psi^j_{2(fk)}) \simeq h^l(\bar{x}^j_{(fk)}) + \bar{X}^2_{(f)}\frac{\partial h^l}{\partial x^j}\psi^j_{2(fk)} \\ y^l_{(mk)} &\simeq h^l(\bar{x}^j_{(mk)} + \bar{X}^2_{(m)}\psi^j_{2(mk)}) \simeq h^l(\bar{x}^j_{(mk)}) + \bar{X}^2_{(m)}\frac{\partial h^l}{\partial x^j}\psi^j_{2(mk)}. \end{aligned} \tag{2.24}$$

Comparing (2.24) with (2.23), we obtain the following transformations for $\bar{x}^l_{(fk)}, \bar{x}^l_{(mk)}$, $\psi^l_{2(fk)}$ and $\psi^l_{2(mk)}$:

$$\begin{aligned} \bar{y}^l_{(fk)} &= h^l(\bar{x}^j_{(fk)}) \\ \bar{y}^l_{(mk)} &= h^l(\bar{x}^j_{(mk)}) \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} \varphi^l_{2(fk)} &= \frac{\partial y^l}{\partial x^j}\psi^j_{2(fk)} \\ \varphi^l_{2(mk)} &= \frac{\partial y^l}{\partial x^j}\psi^j_{2(mk)}. \end{aligned} \tag{2.26}$$

In the sequel it will be found convenient to use vector notation. We introduce in the deformed body a cartesian coordinate system $(z^l)$ which is related to the coordinate system $(x^l)$ by

$$\begin{aligned} z^l &= z^l(x^j) \\ x^j &= x^j(z^l). \end{aligned} \tag{2.27}$$

We define the position vectors $\mathbf{p}_{(fk)}$ and $\mathbf{p}_{(mk)}$ of the $k$th reinforcing and matrix layers, respectively, as

$$\begin{aligned} \mathbf{p}_{(fk)} &= z^l_{(fk)}\mathbf{i}_l; \qquad z^l_{(fk)} = z^l(x_{(fk)}) \\ \mathbf{p}_{(mk)} &= z^l_{(mk)}\mathbf{i}_l; \qquad z^l_{(mk)} = z^l(x_{(mk)}), \end{aligned} \tag{2.28}$$

where $\mathbf{i}_l$ are the unit base vectors of the cartesian system $(z^l)$. We can now express (2.21) in vector notation:

$$\begin{aligned} \mathbf{p}_{(fk)} &\simeq \bar{\mathbf{p}}_{(fk)}(X^1, X^2_{(fk)}, X^3, t) + X^2_{(f)}\boldsymbol{\psi}_{(fk)2}(X^1, X^2_{(fk)}\, X^3, t) \\ \mathbf{p}_{(mk)} &\simeq \bar{\mathbf{p}}_{(mk)}(X^1, X^2_{(mk)}, X^3, t) + X^2_{(m)}\boldsymbol{\psi}_{(mk)2}(X^1, X^2_{(mk)}, X^3, t), \end{aligned} \tag{2.29}$$

where $\bar{\mathbf{p}}_{(fk)}$ and $\bar{\mathbf{p}}_{(mk)}$ are the midsurface position vectors after deformation of the $k$th reinforcing and matrix layers, respectively:

$$\begin{aligned} \bar{\mathbf{p}}_{(fk)} &= \bar{z}^l_{(fk)}\mathbf{i}_l; \qquad \bar{z}^l_{(fk)} = z^l(\bar{x}^j_{(fk)}) \\ \bar{\mathbf{p}}_{(mk)} &= \bar{z}^l_{(mk)}\mathbf{i}_l; \qquad \bar{z}^l_{(mk)} = z^l(\bar{x}^j_{(mk)}), \end{aligned} \tag{2.30}$$

and $\boldsymbol{\psi}_{(fk)2}$ and $\boldsymbol{\psi}_{(mk)2}$ are the local displacements of the reinforcing and matrix layers, respectively:

$$\begin{aligned} \boldsymbol{\psi}_{(fk)2} &= \psi_{(fk)2}{}^l\bar{\mathbf{g}}_{(fk)l} \\ \boldsymbol{\psi}_{(mk)2} &= \psi_{(mk)2}{}^l\bar{\mathbf{g}}_{(mk)l}. \end{aligned} \tag{2.31}$$

In equations (2.22), $\bar{\mathbf{g}}_{(fk)l}$ and $\bar{\mathbf{g}}_{(mk)l}$ are the base vectors $\mathbf{g}_l(x^j)$ of the coordinate system $(x^l)$ evaluated at $\bar{x}^j_{(fk)}$ and $\bar{x}^j_{(mk)}$, respectively:

$$\begin{aligned} \bar{\mathbf{g}}_{(fk)} &\equiv \mathbf{g}_l(\bar{x}^j_{(fk)}) = \frac{\partial z^n}{\partial x^l}(\bar{x}^j_{(fk)})\mathbf{i}_n \\ \bar{\mathbf{g}}_{(mk)} &\equiv \mathbf{g}_l(\bar{x}^j_{(mk)}) = \frac{\partial z^n}{\partial x^l}(\bar{x}^j_{(mk)})\mathbf{i}_n. \end{aligned} \tag{2.32}$$

All the quantities introduced in equations (2.29) or equivalently (2.21) are not independent. The position vector must be continuous at the interfaces of the laminated composite. For the $k$th pair of layers this requires that $\bar{\mathbf{p}}_{(fk)}, \bar{\mathbf{p}}_{(mk)}, \boldsymbol{\psi}_{(fk)2}$ and $\boldsymbol{\psi}_{(mk)2}$ satisfy:

$$\begin{aligned} &\bar{\mathbf{p}}_{(fk)}(X^1, X^2_{(fk)}, X^3, t) - \bar{\mathbf{p}}_{(mk)}(X^1, X^2_{(mk)}, X^3, t) \\ &= \frac{d_f}{2}\boldsymbol{\psi}_{(fk)2}(X^1, X^2_{(fk)}, X^3, t) + \frac{d_m}{2}\boldsymbol{\psi}_{(mk)2}(X^1, X^2_{(mk)}, X^3, t), \end{aligned} \tag{2.33}$$

where $d_f$ and $d_m$ are the thicknesses before deformation of the reinforcing and matrix layers, respectively.

We construct a continuum model of a curvilinear laminated medium by following the same reasoning employed in the case of plane lamination [1–4]. We introduce field variables which are continuous functions of $X^2$ and whose values on the discrete surfaces $X^2 = X^2_{(fk)}$ and $X^2 = X^2_{(mk)}$ are the actual values of the midsurface position vectors and local displacement vectors. This is indicated by writing $\bar{\mathbf{p}}_{(f)}(X^K, t)$ instead of $\bar{\mathbf{p}}_{(fk)}(X^1, X^2_{(fk)}, X^3, t)$, etc. Also, the continuity condition (2.24) suggests that $\bar{\mathbf{p}}_{(f)}$ and $\bar{\mathbf{p}}_{(m)}$ should be represented by the same vector field $\bar{\mathbf{p}}(X^K, t)$, termed the gross position vector, at different locations. By noting that $X^2_{(mk)} = X^2_{(fk)} - \frac{1}{2}(d_m + d_f)$, and by assuming that the thicknesses of the layers are small, the difference relation (2.33) can be replaced by

$$\partial_2\bar{\mathbf{p}}(X^K, t) = \eta\boldsymbol{\psi}_{(f)2}(X^K, t) + (1-\eta)\boldsymbol{\psi}_{(m)}(X^K, t). \tag{2.34}$$

Formally, the passage from (2.33) to (2.34) is justified by taking the limit $d_f \to 0, d_m \to 0$, such that $\eta$, defined by

$$\eta = \frac{d_f}{d_f + d_m}, \tag{2.35}$$

is held constant. It is assumed that the continuity conditions (2.34) hold at each point of the continuum. We are thus replacing the discrete system of parallel layers by a continuum with microstructure.

To write (2.34) in component form we note that the above reasoning is equivalent to replacing $\bar{x}^l_{(f)}$ and $\bar{x}^l_{(m)}$ by $\bar{x}^l$, and $\bar{z}^j_{(f)}$ and $\bar{z}^j_{(m)}$ by $\bar{z}^l$. The gross position vector $\bar{\mathbf{p}}$ is given by

$$\bar{\mathbf{p}} = \bar{z}^j \mathbf{i}_j, \tag{2.36}$$

where $\bar{z}^j$ is related to $\bar{x}^l$ by

$$\begin{aligned} \bar{z}^j &= z^j(\bar{x}^l) \\ \bar{x}^l &= x^l(\bar{z}^j). \end{aligned} \tag{2.37}$$

Introducing the base vectors $\bar{g}_l$

$$\bar{g}_l = \frac{\partial \bar{z}^n}{\partial \bar{x}^l}\mathbf{i}_n, \tag{2.38}$$

we see from (2.36) that

$$\partial_2 \bar{p} = \partial_2 \bar{x}^l \bar{g}_l. \tag{2.39}$$

Also, decomposing $\boldsymbol{\psi}_{(f)2}$ and $\boldsymbol{\psi}_{(m)2}$,

$$\begin{aligned} \boldsymbol{\psi}_{(f)2} &= \psi_{(f)2}{}^l \bar{g}_l \\ \boldsymbol{\psi}_{(m)2} &= \psi_{(m)2}{}^l \bar{g}_l, \end{aligned} \tag{2.40}$$

we obtain from (2.34) that

$$\partial_2 \bar{x}^l(X^K, t) = \eta \psi_{(f)2}{}^l(X^K, t) + (1-\eta)\psi_{(m)2}{}^l(X^K, t). \tag{2.41}$$

In the continuum model of a curvilinear laminated composite, the kinematic variables are $\bar{\mathbf{p}}, \boldsymbol{\psi}_{(f)2}$ and $\boldsymbol{\psi}_{(m)2}$. These variables are not independent but must satisfy the continuity condition (2.34) or equivalently in component form (2.41).

It is often convenient to introduce the gross displacement vector $\bar{\mathbf{U}}$

$$\bar{\mathbf{U}}(X^K, t) = \bar{\mathbf{p}}(X^K, t) - \bar{\mathbf{P}}(X^K) - \mathbf{b}, \tag{2.42}$$

where $\bar{\mathbf{P}}(X^K)$ is the gross position vector before deformation:

$$\bar{\mathbf{P}}(X^K) = \mathbf{Z}(X^K), \tag{2.43}$$

and $\mathbf{b}$ is the vector from the origin of the cartesian system $(Z^K)$ to the origin of the cartesian systems $(z^k)$. If we introduce the components of $\bar{\mathbf{U}}$, $\boldsymbol{\psi}_{(f)2}$, and $\boldsymbol{\psi}_{(m)2}$ in the direction of the base vectors $\bar{\mathbf{G}}_K = \mathbf{G}_K(X^K)$

$$\mathbf{U} = U^K \bar{\mathbf{G}}_K \tag{2.44}$$

$$\boldsymbol{\psi}_{(f)2} = (\psi_{(f)2}{}^K + \delta_2^K)\bar{\mathbf{G}}_K; \qquad \psi_{(f)2}{}^l = \bar{g}^l_K(\psi_{(f)2}{}^K + \delta_2^K) \tag{2.45}$$

$$\boldsymbol{\psi}_{(m)2} = (\psi_{(m)2}{}^K + \delta_2^K)\bar{\mathbf{G}}_K; \qquad \psi_{(m)2}{}^l = \bar{g}^l_K(\psi_{(m)2}{}^K + \delta_2^K), \tag{2.46}$$

where the shifters $\bar{g}^k_K$ are defined by

$$\bar{g}^l_K = \bar{\mathbf{g}}^l \cdot \bar{\mathbf{G}}_K \tag{2.47}$$

and

$$\bar{\mathbf{g}}^l \cdot \bar{\mathbf{g}}_n = \delta^l_n, \tag{2.48}$$

we can express the constraint condition (2.41) as

$$U^K{}_{;2} = \eta \psi_{(f)2}{}^K + (1-\eta)\psi_{(m)2}{}^K, \tag{2.49}$$

where

$$U^K{}_{;L} = \partial_L U^K + \left\{ \begin{matrix} K \\ M \quad L \end{matrix} \right\} U^M. \tag{2.50}$$

## 3. DYNAMICAL BALANCE LAWS

Within each layer the dynamics of the material is fully described by Cauchy's law. For the $k$th pair of reinforcing and matrix layers, this implies that

$$\partial_j \mathbf{t}^j_{(fk)} + \left\{ \begin{matrix} l \\ l \quad j \end{matrix} \right\}_{\mathbf{g}_{(fk)}} \mathbf{t}^j_{(fk)} + \rho_{(fk)} \bar{\mathbf{f}}_{(fk)} = \rho_{(fk)} \mathbf{a}_{(fk)} \tag{3.1}$$

$$\partial_j \mathbf{t}^j_{(fk)} + \left\{ \begin{matrix} l \\ l \quad j \end{matrix} \right\}_{\mathbf{g}_{(mk)}} \mathbf{t}^j_{(mk)} + \rho_{(mk)} \mathbf{f}_{(mk)} = \rho_{(mk)} \mathbf{a}_{(mk)}, \tag{3.2}$$

where $\mathbf{t}^j_{(fk)}$ and $\mathbf{t}^j_{(mk)}$ are the stress vectors acting on the $j$th coordinate surface of the $k$th reinforcing and $k$th matrix layer, respectively; $\mathbf{f}_{(fk)}$ and $\mathbf{f}_{(mk)}$ their respective body force vectors; $\rho_{(fk)}$ and $\rho_{(mk)}$ their respective mass densities in the deformed body; and $\mathbf{a}_{(fk)}$ and $\mathbf{a}_{(mk)}$ their respective accelerations. If we introduce the Piola stress vectors [6, 7]

$$\begin{aligned} \mathbf{T}^K_{(fk)} &= J_{(fk)} \partial_j X^K_{(fk)} \mathbf{t}^j_{(fk)}; \qquad J_{(fk)} = \det(\partial_K x^i_{(fk)}) \\ \mathbf{T}^K_{(mk)} &= J_{(mk)} \partial_j X^K_{(mk)} \mathbf{t}^j_{(mk)}; \qquad J_{(mk)} = \det(\partial_K x^i_{(mk)}) \end{aligned} \tag{3.3}$$

and the undeformed densities†

$$\begin{aligned} \rho_f &= \rho_{(fk)} J_{(fk)} \\ \rho_m &= \rho_{(mk)} J_{(mk)}, \end{aligned} \tag{3.4}$$

we can write the balance laws (3.1) and (3.2) in the form [6, 7]

$$\partial_K \mathbf{T}^K_{(fk)} + \left\{ \begin{matrix} L \\ L \quad K \end{matrix} \right\}_{\mathbf{G}_{(fk)}} \mathbf{T}^K_{(fk)} + \rho_f \mathbf{f}_{(fk)} = \rho_f \mathbf{a}_{(fk)} \tag{3.5}$$

$$\partial_K \mathbf{T}^K_{(mk)} + \left\{ \begin{matrix} L \\ L \quad K \end{matrix} \right\}_{\mathbf{G}_{(mk)}} \mathbf{T}^K_{(mk)} + \rho_m \mathbf{f}_{(mk)} = \rho_m \mathbf{a}_{(mk)}. \tag{3.6}$$

† We assume that $\rho_f$ and $\rho_m$ are constants, i.e. the layers are homogeneous.

Introducing the relative coordinate systems at the midsurface of each layer, we can express (3.5) and (3.6) as

$$\partial_\alpha(K^*_{(fk)}\mathbf{T}^\alpha_{(fk)})+K^*_{(fk)}\mathbf{T}^\alpha_{(fk)}\left\{\begin{matrix}\beta\\ \beta\ \ \alpha\end{matrix}\right\}_{\bar{\mathbf{G}}_{(fk)}}+\partial_2(K^*_{(fk)}\mathbf{T}^2_{(fk)})$$
$$+\rho_f K^*_{(fk)}\mathbf{f}_{(fk)} = \rho_f K^*_{(fk)}\mathbf{a}_{(fk)} \tag{3.7}$$

$$\partial_\alpha(K^*_{(mk)}\mathbf{T}^\alpha_{(mk)})+K^*_{(mk)}\mathbf{T}^\alpha_{(mk)}\left\{\begin{matrix}\beta\\ \beta\ \ \alpha\end{matrix}\right\}_{\bar{\mathbf{G}}_{(mk)}}+\partial_2(K^*_{(mk)}\mathbf{T}^2_{(mk)})$$
$$+\rho_m K^*_{(mk)}\mathbf{f}_{(mk)} = \rho_m K^*_{(mk)}\mathbf{a}_{(mk)}, \tag{3.8}$$

where $\bar{\mathbf{G}}_{(fk)}$ and $\bar{\mathbf{G}}_{(mk)}$ are the metric tensors of the midsurface of the $k$th fiber and matrix layers, respectively; $K^*_{(fk)}$ and $K^*_{(mk)}$ are defined by (2.19). In deriving (3.7) and (3.8) we have used the identities (2.11).

To obtain the balance laws of a laminated composite we integrate (3.7) and (3.8) over the $k$th pair of layers:

$$\partial_\alpha\bar{\mathbf{T}}^\alpha_{(k)}+\frac{1}{d_f+d_m}\operatorname{int}^{(fk)}(K^*_{(fk)}\mathbf{T}^\alpha_{(fk)})\left\{\begin{matrix}\beta\\ \beta\ \ \alpha\end{matrix}\right\}_{\bar{\mathbf{G}}_{(fk)}}$$
$$+\frac{1}{d_f+d_m}\operatorname{int}^{(mk)}(K^*_{(mk)}\mathbf{T}^\alpha_{(mk)})\left\{\begin{matrix}\beta\\ \beta\ \ \alpha\end{matrix}\right\}_{\bar{\mathbf{G}}_{(mk)}}$$
$$+\frac{1}{d_f+d_m}\{[K^*_{(fk)}\mathbf{T}^2_{(fk)}]_{\bar{X}^2_{(f)}=\frac{1}{2}d_f}-[K^*_{(fk)}\mathbf{T}^2_{(fk)}]_{\bar{X}^2_{(f)}=-\frac{1}{2}d_f}$$
$$+[K^*_{(mk)}\mathbf{T}^2_{(mk)}]_{\bar{X}^2_{(m)}=\frac{1}{2}d_m}-[K^*_{(mk)}\mathbf{T}^2_{(mk)}]_{\bar{X}^2_{(m)}=-\frac{1}{2}d_m}\}+\rho\bar{\mathbf{f}}_{(k)} = \rho\bar{\mathbf{a}}_{(k)}, \tag{3.9}$$

where we have defined

$$\bar{\mathbf{T}}^\alpha_{(k)} = \frac{1}{d_f+d_m}\operatorname{int}^{(fk)}(K^*_{(fk)}\mathbf{T}^\alpha_{(fk)})+\frac{1}{d_f+d_m}\operatorname{int}^{(mk)}(K^*_{(mk)}\mathbf{T}^\alpha_{(mk)}) \tag{3.10}$$

$$\rho\bar{\mathbf{f}}_{(k)} = \frac{1}{d_f+d_m}\operatorname{int}^{(fk)}(\rho_f K^*_{(fk)}\mathbf{f}_{(fk)})+\frac{1}{d_f+d_m}\operatorname{int}^{(mk)}(\rho_m K^*_{(mk)}\mathbf{f}_{(mk)}) \tag{3.11}$$

$$\rho\bar{\mathbf{a}}_{(k)} = \frac{1}{d_f+d_m}\operatorname{int}^{(fk)}(\rho_f K^*_{(fk)}\mathbf{a}_{(mk)})+\frac{1}{d_f+d_m}\operatorname{int}^{(mk)}(\rho_m K^*_{(mk)}\mathbf{a}_{(mk)}), \tag{3.12}$$

where

$$\rho = \eta\rho_f+(1-\eta)\rho_m. \tag{3.13}$$

In the above equations we have found it convenient to use the following notation to indicate integration over the $k$th reinforcing layer of a function $g_{(fk)}(X^1, X^2_{(fk)}, \bar{X}^2_{(f)}, X^3, t)$:

$$\operatorname{int}^{(fk)}(g_{(fk)}) \equiv \int_{-\frac{1}{2}d_f}^{\frac{1}{2}d_f} g_{(fk)}(X^1, X^2_{(fk)}, \bar{X}^2_{(f)}, X^3, t)\,\mathrm{d}\bar{X}^2_{(f)}, \tag{3.14}$$

and similarly for integration over the $k$th matrix layer.

The passage to the continuum model is made by introducing field variables $\bar{\mathbf{T}}^\alpha(X^K, t)$, $\bar{\mathbf{f}}(X^K, t)$ and $\bar{\mathbf{a}}(X^K, t)$, which at some point $X^2$ within the $k$th pair of layers assume the values $\mathbf{T}^\alpha_{(k)}$, $\bar{\mathbf{f}}_{(k)}$ and $\bar{\mathbf{a}}_{(k)}$. Also, assuming that

$$\frac{d_f}{R} \leq \leq 1, \qquad \frac{d_m}{R} \leq < 1, \tag{3.15}$$

where $R$ is the minimum radius of curvature of the midsurfaces, we can write in the first approximation

$$\begin{Bmatrix} \beta \\ \beta \ \alpha \end{Bmatrix}_{\bar{\mathbf{G}}_{(fk)}} \simeq \begin{Bmatrix} \beta \\ \beta \ \alpha \end{Bmatrix}_{\bar{\mathbf{G}}_{(mk)}} \simeq \begin{Bmatrix} \beta \\ \beta \ \alpha \end{Bmatrix}_{\bar{\mathbf{G}}}, \tag{3.16}$$

where $\bar{\mathbf{G}}$ is the metric tensor at some point $X^K$ in the $k$th pair of layers. Consistent with (3.15), in the integrals (3.10)–(3.12), we can set

$$K^*_{(fk)} \approx K^*_{(mk)} \approx 1. \tag{3.17}$$

For the interfacial stress vector we introduce a vector function $\boldsymbol{\Sigma}^2(X^K, t)$ which on the discrete interfaces of the lamination coincides with the interfacial stresses. Using the continuity of stress at the interfaces we can write the expression in brackets in (3.9) as

$$\begin{aligned} &\frac{1}{d_f+d_m}\{[K^*_{(fk)}\mathbf{T}^2_{(fk)}]_{\bar{X}^2_{(f)}=\frac{1}{2}d_f} - [K^*_{(fk)}\mathbf{T}^2_{(fk)}]_{\bar{X}^2_{(f)}=-\frac{1}{2}d_f} \\ &\quad + [K^*_{(mk)}\mathbf{T}^2_{(mk)}]_{\bar{X}^2_{(m)}=\frac{1}{2}d_m} - [K^*_{(mk)}\mathbf{T}^2_{(mk)}]_{\bar{X}^2_{(m)}=-\frac{1}{2}d_m}\} \\ &= \frac{1}{d_f+d_m}\{K^*_{(fk)}(\tfrac{1}{2}d_f)\boldsymbol{\Sigma}^2(X^2+d_f) + [K^*_{(mk)}(\tfrac{1}{2}d_m) \\ &\quad - K^*_{(fk)}(-\tfrac{1}{2}d_f)]\boldsymbol{\Sigma}^2(X^2) - K^*_{(mk)}(-\tfrac{1}{2}d_m)\boldsymbol{\Sigma}^2(X^2-d_m)\}, \end{aligned} \tag{3.18}$$

which in the first approximation can be replaced by

$$\approx \partial_2\boldsymbol{\Sigma}^2 + \begin{Bmatrix} K \\ K \ \ 2 \end{Bmatrix}_{\bar{\mathbf{G}}} \boldsymbol{\Sigma}^2, \tag{3.19}$$

where we have used (2.11).

Similarly, from (3.12) and (2.29), to within first order terms, the average acceleration $\bar{\mathbf{a}}(X^K, t)$ can be written as

$$\bar{\mathbf{a}}(X^K, t) = \rho\ddot{\bar{\mathbf{p}}}(X^K, t). \tag{3.20}$$

Thus the balance of linear momentum for a curvilinear laminated composite can be expressed as

$$\partial_\alpha\mathbf{T}^\alpha + \bar{\mathbf{T}}^\alpha\begin{Bmatrix} \beta \\ \beta \ \alpha \end{Bmatrix}_{\bar{\mathbf{G}}} + \partial_2\boldsymbol{\Sigma}^2 + \boldsymbol{\Sigma}^2\begin{Bmatrix} \beta \\ \beta \ \ 2 \end{Bmatrix}_{\bar{\mathbf{G}}} + \rho\bar{\mathbf{f}} = \rho\ddot{\bar{\mathbf{p}}}. \tag{3.21}$$

The remaining dynamical balance laws for a curvilinear laminated composite may be derived by multiplying (3.7) by $\bar{X}^2_{(f)}$ and (3.8) by $\bar{X}^2_{(m)}$ and integrating over the respective layer thicknesses. For the reinforcing layer we obtain

$$\partial_\alpha\mathbf{M}^{\alpha 2}_{(fk)} + \mathbf{M}^{\alpha 2}_{(fk)}\begin{Bmatrix} \beta \\ \beta \ \alpha \end{Bmatrix}_{\bar{\mathbf{G}}_{(fk)}} + \frac{1}{d_f}\operatorname{int}^{(fk)}\left(\bar{X}^2_{(f)}\frac{\partial(K^*_{(fk)}\mathbf{T}^2_{(fk)})}{\partial\bar{X}^2_{(f)}}\right) + \rho_f\mathbf{l}^2_{(fk)} = \rho_f\boldsymbol{\omega}^2_{(fk)}, \tag{3.22}$$

where we have defined

$$\mathbf{M}^{\alpha 2}_{(fk)} = \frac{1}{d_f}\operatorname{int}^{(fk)}(\bar{X}^2_{(f)}K^*_{(fk)}\mathbf{T}^{\alpha}_{(fk)}) \tag{3.23}$$

$$\mathbf{l}^2_{(fk)} = \frac{1}{d_f}\operatorname{int}^{(fk)}(\bar{X}^2_{(f)}K^*_{(fk)}\mathbf{f}_{(fk)}) \tag{3.24}$$

$$\boldsymbol{\omega}^2_{(fk)} = \frac{1}{d_f}\operatorname{int}^{(fk)}(\bar{X}^2_{(f)}K^*_{(fk)}\mathbf{a}_{(fk)}). \tag{3.25}$$

Integration by parts of the integral appearing in (3.22) yields

$$\frac{1}{d_f}\operatorname{int}^{(fk)}\left(\bar{X}^2_{(f)}\frac{\partial(K^*_{(fk)}\mathbf{T}^2_{(fk)})}{\partial\bar{X}^2_{(f)}}\right) = \tfrac{1}{2}\{[K^*_{(fk)}\mathbf{T}^2_{(fk)}]_{\bar{X}^2_{(f)}=\frac{1}{2}d_f}+[K^*_{(fk)}\mathbf{T}^2_{(fk)}]_{\bar{X}^2_{(f)}=-\frac{1}{2}d_f}\}-\bar{\mathbf{T}}^2_{(fk)}, \tag{3.26}$$

where

$$\bar{\mathbf{T}}^2_{(fk)} = \frac{1}{d_f}\operatorname{int}^{(fk)}(K^*_{(fk)}\mathbf{T}^2_{(fk)}). \tag{3.27}$$

In the continuum model of a laminated medium the term in brackets in (3.26) can be replaced by $\boldsymbol{\Sigma}^2(X^K, t)$; thus (3.26) reduces to

$$\frac{1}{d_f}\operatorname{int}^{(fk)}\left(\bar{X}^2_{(f)}\frac{\partial(K^*_{(fk)}\mathbf{T}^2_{(fk)})}{\partial\bar{X}^2_{(f)}}\right) \approx \boldsymbol{\Sigma}^2-\bar{\mathbf{T}}^2_{(f)}, \tag{3.28}$$

where we introduce a field quantity $\bar{\mathbf{T}}^2_{(f)}(X^K, t)$ which at some point $X^2$ within the $k$th reinforcing layer assumes the value $\bar{\mathbf{T}}^2_{(fk)}$. Similarly, we replace the quantities $\mathbf{M}^{\alpha 2}_{(fk)}$, $\mathbf{l}^2_{(fk)}$ and $\boldsymbol{\omega}^2_{(fk)}$ by vector fields $\mathbf{M}^{\alpha 2}_{(f)}(X^K, t)$, $\mathbf{l}^2_{(f)}(X^K, t)$ and $\boldsymbol{\omega}^2_{(f)}(X^K, t)$, respectively. Using (2.29) and (3.15) in (3.25), in the continuum model $\boldsymbol{\omega}^2_{(f)}$ has the form

$$\boldsymbol{\omega}^2_{(f)}(X^K, t) = \rho_f J_f \ddot{\boldsymbol{\psi}}_{(f)2}, \tag{3.29}$$

where

$$J_f = \tfrac{1}{12}(d_f)^2. \tag{3.30}$$

Substituting (3.28) and (3.29) in (3.22) we see that, in the continuum model, the balance of moment of momentum for the reinforcing layer becomes

$$\partial_\alpha \mathbf{M}^{\alpha 2}_{(f)}+\mathbf{M}^{\alpha 2}_{(f)}\begin{Bmatrix}\beta\\ \beta\ \ \alpha\end{Bmatrix}_{\bar{G}}+\boldsymbol{\Sigma}^2-\bar{\mathbf{T}}^2_{(f)}+\rho_f\mathbf{l}^2_{(f)} = \rho_f J_f \boldsymbol{\psi}_{(f)2}. \tag{3.31}$$

Similarly, for the matrix, we deduce that

$$\partial_\alpha \mathbf{M}^{\alpha 2}_{(m)}+\mathbf{M}^{\alpha 2}_{(m)}\begin{Bmatrix}\beta\\ \beta\ \ \alpha\end{Bmatrix}_{\bar{G}}+\boldsymbol{\Sigma}^2-\bar{\mathbf{T}}^2_{(m)}+\rho_m\mathbf{l}^2_{(f)} = \rho_m J_m \boldsymbol{\psi}_{(m)2}. \tag{3.32}$$

By using the identity

$$\frac{1}{\sqrt{(G)}}\partial_K\sqrt{(G)} = \begin{Bmatrix}L\\ L\ \ K\end{Bmatrix} \tag{3.33}$$

and (2.11), equations (3.21), (3.31) and (3.32) can be put in the form

$$\frac{1}{\sqrt{(G)}}\partial_\alpha(\sqrt{(G)}\bar{\mathbf{T}}^\alpha)+\frac{1}{\sqrt{(G)}}\partial_2(\sqrt{(G)}\boldsymbol{\Sigma}^2)+\rho\mathbf{f} = \rho\ddot{\mathbf{p}} \tag{3.34}$$

$$\frac{1}{\sqrt{(G)}}\partial_\alpha(\sqrt{(G)}\mathbf{M}^{\alpha 2}_{(f)})+\boldsymbol{\Sigma}^2-\bar{\mathbf{T}}^2_{(f)}+\rho_f\mathbf{l}^2_{(f)} = \rho_f I_f\ddot{\boldsymbol{\Psi}}_{(f)2} \tag{3.35}$$

$$\frac{1}{\sqrt{(G)}}\partial_\alpha(\sqrt{(G)}\mathbf{M}^{\alpha 2}_{(m)})+\boldsymbol{\Sigma}^2-\bar{\mathbf{T}}^2_{(m)}+\rho_m\mathbf{l}^2_{(m)} = \rho_m I_m\ddot{\boldsymbol{\Psi}}_{(m)2}. \tag{3.36}$$

If we choose a coordinate system $x^l$ of the deformed body with base vectors $\bar{\mathbf{g}}_l$, we can write (3.34)–(3.36) in component form:

$$\begin{aligned}\partial_\alpha\bar{T}^{\alpha l}+\bar{T}^{\alpha l}\left\{\begin{matrix}\beta\\ \beta\ \ \alpha\end{matrix}\right\}_{\bar{\mathbf{G}}}+\bar{T}^{\alpha n}\left\{\begin{matrix}l\\ n\ \ j\end{matrix}\right\}_{\bar{\mathbf{g}}}\partial_\alpha\bar{x}^j+\partial_2\Sigma^{2l}+\Sigma^{2l}\left\{\begin{matrix}\beta\\ \beta\ \ 2\end{matrix}\right\}_{\bar{\mathbf{G}}}\\ +\Sigma^{2n}\left\{\begin{matrix}l\\ n\ \ j\end{matrix}\right\}_{\bar{\mathbf{g}}}\partial_2\bar{x}^j+\rho\bar{f}^l=\rho\ddot{\bar{x}}^l\end{aligned} \tag{3.37}$$

$$\begin{aligned}\partial_\alpha M^{\alpha 2l}_{(f)}+M^{\alpha 2l}_{(f)}\left\{\begin{matrix}\beta\\ \beta\ \ \alpha\end{matrix}\right\}_{\bar{\mathbf{G}}}+M^{\alpha 2n}_{(f)}\left\{\begin{matrix}l\\ n\ \ j\end{matrix}\right\}_{\bar{\mathbf{g}}}\partial_\alpha\bar{x}^j+\Sigma^{2l}-\bar{T}^{2l}_{(f)}\\ +\rho_f l^{2l}_{(f)} = \rho_f J_f\ddot{\psi}_{(f)2}{}^l\end{aligned} \tag{3.38}$$

$$\begin{aligned}\partial_\alpha M^{\alpha 2l}_{(m)}+M^{\alpha 2l}_{(m)}\left\{\begin{matrix}\beta\\ \beta\ \ \alpha\end{matrix}\right\}_{\bar{\mathbf{G}}}+M^{\alpha 2n}_{(m)}\left\{\begin{matrix}l\\ n\ \ j\end{matrix}\right\}_{\bar{\mathbf{g}}}\partial_\alpha\bar{x}^j+\Sigma^{2l}-\bar{T}^{2l}_{(m)}\\ +\rho_m l^{2l}_{(m)} = \rho_m J_m\ddot{\psi}_{(m)2}{}^l\end{aligned} \tag{3.39}$$

where

$$\begin{aligned}\bar{\mathbf{T}}^\alpha &= \bar{T}^{\alpha l}\bar{\mathbf{g}}_l\\ \boldsymbol{\Sigma}^2 &= \Sigma^{2l}\bar{\mathbf{g}}_l\\ \mathbf{M}^{\alpha 2}_{(f)} &= M^{\alpha 2l}_{(f)}\bar{\mathbf{g}}_l\\ \mathbf{M}^{\alpha 2}_{(m)} &= M^{\alpha 2l}_{(m)}\bar{\mathbf{g}}_l\\ \bar{\mathbf{T}}^2_{(f)} &= \bar{T}^{2l}_{(f)}\bar{\mathbf{g}}_l\\ \bar{\mathbf{T}}^2_{(m)} &= \bar{T}^{2l}_{(m)}\bar{\mathbf{g}}_l\\ \bar{\mathbf{f}} &= \bar{f}^l\bar{\mathbf{g}}_l\\ \mathbf{l}^2_{(f)} &= l^{2l}_{(f)}\bar{\mathbf{g}}_l\\ \mathbf{l}^2_{(m)} &= l^{2l}_{(m)}\bar{\mathbf{g}}_l.\end{aligned} \tag{3.40}$$

Similarly, the balance laws (3.21), (3.31) and (3.32) can be expressed in component form in the system $(X^K)$:

$$\partial_\alpha \bar{T}^{\alpha K} + \bar{T}^{\alpha K}\begin{Bmatrix} L \\ L \quad \alpha \end{Bmatrix}_{\bar{\mathbf{G}}} + \bar{T}^{\alpha N}\begin{Bmatrix} K \\ N \quad \alpha \end{Bmatrix}_{\bar{\mathbf{G}}} + \partial_2 \Sigma^{2K}\begin{Bmatrix} L \\ L \quad 2 \end{Bmatrix}_{\bar{\mathbf{G}}} + \Sigma^{2N}\begin{Bmatrix} K \\ N \quad 2 \end{Bmatrix}_{\bar{\mathbf{G}}} + \rho \bar{f}^K = \rho \ddot{\bar{U}}^K \tag{3.41}$$

$$\partial_\alpha M_{(f)}^{\alpha 2K} + M_{(f)}^{\alpha 2K}\begin{Bmatrix} L \\ L \quad \alpha \end{Bmatrix}_{\bar{\mathbf{G}}} + M_{(f)}^{\alpha 2N}\begin{Bmatrix} K \\ N \quad \alpha \end{Bmatrix}_{\bar{\mathbf{G}}} + \Sigma^{2K} - \bar{T}_{(f)}^{2K} + \rho_f l_{(f)}^{2K} = \rho_f J_f \ddot{\psi}_{(f)2}{}^K \tag{3.42}$$

$$\partial_\alpha M_{(m)}^{\alpha 2K} + M_{(m)}^{\alpha 2K}\begin{Bmatrix} L \\ L \quad \alpha \end{Bmatrix}_{\bar{\mathbf{G}}} + M_{(m)}^{\alpha 2N}\begin{Bmatrix} K \\ N \quad \alpha \end{Bmatrix}_{\bar{\mathbf{G}}} + \Sigma^{2K} - \bar{T}_{(m)}^{2K} + \rho_m l_{(m)}^{2K} = \rho_m J_m \ddot{\psi}_{(m)2}{}^K, \tag{3.43}$$

where

$$\begin{aligned}
\mathbf{T}^\alpha &= \bar{T}^{\alpha K}\bar{\mathbf{G}}_K, & \bar{T}^{\alpha l} &= \bar{g}^l{}_K \bar{T}^{\alpha K} \\
\mathbf{\Sigma}^2 &= \Sigma^{2K}\bar{\mathbf{G}}_K, & \Sigma^{2l} &= \bar{g}^l{}_K \Sigma^{2K} \\
\mathbf{M}_{(f)}^{\alpha 2} &= M_{(f)}^{\alpha 2K}\bar{\mathbf{G}}_K, & M_{(f)}^{\alpha 2l} &= \bar{g}^l{}_K M_{(f)}^{\alpha 2K} \\
\mathbf{M}_{(m)}^{\alpha 2} &= M_{(m)}^{\alpha 2K}\bar{\mathbf{G}}_K, & M_{(m)}^{\alpha 2l} &= \bar{g}^l{}_K M_{(m)}^{\alpha 2K} \\
\bar{\mathbf{T}}_{(f)}^2 &= \bar{T}_{(f)}^{2K}\bar{\mathbf{G}}_K, & \bar{T}_{(f)}^{2l} &= \bar{g}^l{}_K \bar{T}_{(f)}^{2K} \\
\bar{\mathbf{T}}_{(m)}^2 &= \bar{T}_{(m)}^{2K}\bar{\mathbf{G}}_K, & \bar{T}_{(m)}^{2l} &= \bar{g}^l{}_K \bar{T}_{(m)}^{2K} \\
\bar{\mathbf{f}} &= \bar{f}^K\bar{\mathbf{G}}_K, & \bar{f}^l &= \bar{g}^l{}_K \bar{f}^K \\
\mathbf{l}_{(f)}^2 &= l_{(f)}^{2K}\bar{\mathbf{G}}_K, & l_{(f)}^{2l} &= \bar{g}^l{}_K l_{(f)}^{2K} \\
\mathbf{l}_{(m)}^2 &= l_{(m)}^{2K}\bar{\mathbf{G}}_K, & l_{(m)}^{2l} &= \bar{g}^l{}_K l_{(m)}^{2K},
\end{aligned} \tag{3.44}$$

and $\bar{g}^l{}_K$ is defined by (2.47).

The system of equations (3.21), (3.31) and (3.32), or the equivalent forms (3.34)–(3.36), (3.37)–(3.39) or (3.41)–(3.43), form the basic balance laws in the continuum model for a curvilinear laminated composite. The boundary conditions corresponding to the balance laws are

$$\begin{aligned}
\bar{\mathbf{T}}^\alpha N_\alpha + \mathbf{\Sigma}^2 N_2 &= \mathbf{H} \\
\mathbf{M}_{(f)}^{\alpha 2} N_\alpha &= \mathbf{H}_{(f)}^2 \\
\mathbf{M}_{(m)}^{\alpha 2} N_\alpha &= \mathbf{H}_{(m)}^2,
\end{aligned} \tag{3.45}$$

where $\mathbf{N}$ is the normal before deformation of the boundary

$$\mathbf{N} = N^\alpha \mathbf{G}_\alpha + N^2 \mathbf{G}_2$$

and $\mathbf{H}$, $\mathbf{H}_{(f)}^2$, $\mathbf{H}_{(m)}^2$ are specified vector functions on the boundary. See [3] for a discussion of these boundary conditions.

## 4. CONSTITUTIVE EQUATIONS FOR ELASTIC CURVILINEAR LAMINATED COMPOSITES

To the kinematical and dynamical equations of a composite material formulated in Sections 2 and 3 must be added the constitutive equations relating the stresses and stress-moments to the deformation. In principle these constitutive equations in the continuum model can be derived from equations (3.10), (3.23) and (3.27), when the constitutive relations of the reinforcing and matrix layers are specified. In this section we present the constitutive relations for a laminated composite whose layers are elastic. Thus, within the $k$th pair of layers the stress–deformation relations are

$$\begin{aligned} \mathbf{T}^K_{(fk)} &= \rho_f \frac{\partial F_{(fk)}}{\partial(\partial_K \mathbf{p}_{(fk)})}, \qquad F_{(fk)} = F_f(\partial_K \mathbf{p}_{(fk)}) \\ \mathbf{T}^K_{(mk)} &= \rho_m \frac{\partial F_{(mk)}}{\partial(\partial_K \mathbf{p}_{(mk)})}, \qquad F_{(mk)} = F_m(\partial_K \mathbf{p}_{(mk)}), \end{aligned} \tag{4.1}$$

where $F_f$ and $F_m$ are the stress potentials of the reinforcing and matrix layers, respectively.† From equations (2.29) we have that

$$\begin{aligned} \partial_\alpha \mathbf{p}_{(fk)} &\simeq \partial_\alpha \bar{\mathbf{p}}_{(fk)} + \bar{X}^2_{(f)} \partial_\alpha \boldsymbol{\psi}_{(fk)2} \\ \partial_2 \mathbf{p}_{(fk)} &\simeq \boldsymbol{\psi}_{(fk)2} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \partial_\alpha \mathbf{p}_{(mk)} &\simeq \partial_\alpha \bar{\mathbf{p}}_{(mk)} + \bar{X}^2_{(m)} \partial_\alpha \boldsymbol{\psi}_{(fk)2} \\ \partial_2 \mathbf{p}_{(mk)} &\simeq \boldsymbol{\psi}_{(mk)2}. \end{aligned} \tag{4.3}$$

We introduce the stress potential $F_{(k)}$ of the $k$th pair of layers as

$$\begin{aligned} \rho \bar{F}_{(k)} &= \frac{1}{d_f + d_m} \rho_f \operatorname{int}^{(fk)}[K^*_{(fk)} F_f(\partial_\alpha \bar{\mathbf{p}}_{(fk)} + \bar{X}^2_{(f)} \partial_\alpha \boldsymbol{\psi}_{(fk)2}, \boldsymbol{\psi}_{(fk)2})] \\ &\quad + \frac{1}{d_f + d_m} \rho_m \operatorname{int}^{(mk)}[K^*_{(mk)} F_m(\partial_\alpha \bar{\mathbf{p}}_{(mk)} + \bar{X}^2_{(m)} \partial_\alpha \boldsymbol{\psi}_{(mk)2}, \boldsymbol{\psi}_{(mk)2})], \end{aligned} \tag{4.4}$$

where we have used (4.2) and (4.3) in (4.1). From (3.10), (3.23), (3.27), (4.1) and (4.4), we obtain

$$\bar{\mathbf{T}}^\alpha_{(k)} = \rho \frac{\partial \bar{F}_{(k)}}{\partial(\partial_\alpha \bar{\mathbf{p}}_{(fk)})} + \rho \frac{\partial \bar{F}_{(k)}}{\partial(\partial_\alpha \bar{\mathbf{p}}_{(mk)})} \tag{4.5}$$

$$\eta \mathbf{M}^{\alpha 2}_{(fk)} = \rho \frac{\partial \bar{F}_{(k)}}{\partial(\partial_\alpha \boldsymbol{\psi}_{(fk)2})} \tag{4.6}$$

$$(1-\eta) \mathbf{M}^{\alpha 2}_{(mk)} = \rho \frac{\partial \bar{F}_{(k)}}{\partial(\partial_\alpha \boldsymbol{\psi}_{(mk)2})} \tag{4.7}$$

$$\eta \bar{\mathbf{T}}^2_{(fk)} = \rho \frac{\partial \bar{F}_{(k)}}{\partial \boldsymbol{\psi}_{(fk)2}} \tag{4.8}$$

$$(1-\eta) \bar{T}^2_{(mk)} = \rho \frac{\partial \bar{F}_{(k)}}{\partial \boldsymbol{\psi}_{(mk)2}}. \tag{4.9}$$

† We suppress the dependence of the constitutive functions $F_f$ and $F_m$ on the matrix tensor $G_{KL}$ and various director fields $\mathbf{D}^{(a)}$ which define the symmetry properties of the material.

Following the reasoning of Section 3, to obtain the constitutive relations in the continuum model of an elastic laminated composite we define $\bar{F}$:

$$\rho\bar{F}(\partial_\alpha\bar{\mathbf{p}}, \partial_\alpha\boldsymbol{\psi}_{(f)2}, \partial_\alpha\boldsymbol{\psi}_{(m)2}, \boldsymbol{\psi}_{(f)2}, \boldsymbol{\psi}_{(m)2})$$

$$= \frac{1}{d_f+d_m}\int_{-\frac{1}{2}d_f}^{+\frac{1}{2}d_f} \rho_f F_f(\partial_\alpha\bar{\mathbf{p}} + \bar{X}^2_{(f)}\partial_\alpha\boldsymbol{\psi}_{(f)2}, \boldsymbol{\psi}_{(f)2})\,\mathrm{d}\bar{X}^2_{(f)}$$

$$+ \frac{1}{d_f+d_m}\int_{-\frac{1}{2}d_m}^{+\frac{1}{2}d_m} \rho_m F_m(\partial_\alpha\bar{\mathbf{p}} + \bar{X}^2_{(m)}\partial_\alpha\boldsymbol{\psi}_{(m)2}, \boldsymbol{\psi}_{(m)2})\,\mathrm{d}\bar{X}^2_{(m)}. \tag{4.10}$$

In the continuum model the constitutive equations (4.5)–(4.9) thus reduce to

$$\bar{\mathbf{T}}^\alpha = \rho\frac{\partial\bar{F}}{\partial(\partial_\alpha\bar{\mathbf{p}})} \tag{4.11}$$

$$\eta\mathbf{M}^{\alpha 2}_{(f)} = \rho\frac{\partial\bar{F}}{\partial(\partial_\alpha\boldsymbol{\psi}_{(f)2})} \tag{4.12}$$

$$(1-\eta)\mathbf{M}^{\alpha 2}_{(m)} = \rho\frac{\partial\bar{F}}{\partial(\partial_\alpha\boldsymbol{\psi}_{(m)2})} \tag{4.13}$$

$$\eta\bar{\mathbf{T}}^2_{(f)} = \rho\frac{\partial\bar{F}}{\partial\boldsymbol{\psi}_{(f)2}} \tag{4.14}$$

$$(1-\eta)\bar{\mathbf{T}}^2_{(m)} = \rho\frac{\partial\bar{F}}{\partial\boldsymbol{\psi}_{(m)2}}. \tag{4.15}$$

In component form equations (4.11)–(4.15) become

$$\bar{T}^\alpha{}_l = \rho\frac{\partial\bar{F}}{\partial(\partial_\alpha\bar{x}^l)} \tag{4.16}$$

$$\eta M_{(f)}{}^{\alpha 2}{}_l = \rho\frac{\partial\bar{F}}{\partial(\psi_{(f)2}{}^l{}_{;\alpha})} \tag{4.17}$$

$$(1-\eta)M_{(m)}{}^{\alpha 2}{}_l = \rho\frac{\partial\bar{F}}{\partial(\psi_{(m)2}{}^l{}_{;\alpha})} \tag{4.18}$$

$$\eta\bar{T}_{(f)}{}^2{}_l = \rho\frac{\partial\bar{F}}{\partial\psi_{(f)2}{}^l} \tag{4.19}$$

$$(1-\eta)\bar{T}_{(m)}{}^2{}_l = \rho\frac{\partial\bar{F}}{\partial\psi_{(m)2}{}^l}, \tag{4.20}$$

where we have employed

$$\begin{aligned}\partial_\alpha\bar{\mathbf{p}} &= (\partial_\alpha\bar{x}^l)\bar{\mathbf{g}}_l\\ \partial_\alpha\boldsymbol{\psi}_{(f)2} &= (\psi_{(f)2}{}^l{}_{;\alpha})\bar{\mathbf{g}}_l\\ \partial_\alpha\boldsymbol{\psi}_{(m)2} &= (\psi_{(m)2}{}^l{}_{;\alpha})\bar{\mathbf{g}}_l.\end{aligned} \tag{4.21}$$

In (4.21), we have introduced the total covariant derivative

$$\begin{aligned}\psi_{(f)2}{}^{l}{}_{;\alpha} &= \partial_\alpha \psi_{(f)2}{}^{l} + \begin{Bmatrix} l \\ j\ n \end{Bmatrix} \psi_{(f)2}{}^{n} \partial_\alpha \bar{x}^{j} \\ \psi_{(m)2}{}^{l}{}_{;\alpha} &= \partial_\alpha \psi_{(m)2}{}^{l} + \begin{Bmatrix} l \\ j\ n \end{Bmatrix} \psi_{(m)2}{}^{n} \partial_\alpha \bar{x}^{j}.\end{aligned} \tag{4.22}$$

From (2.42) we can show that

$$\bar{x}^{l}{}_{,K} = \bar{g}^{l}{}_{L}(\delta^{L}{}_{K} + U^{L}{}_{;K}). \tag{4.23}$$

Using (3.44), (4.16)–(4.20) and (4.23), we obtain

$$\bar{T}^{\alpha K} = \rho \frac{\partial \bar{F}}{\partial (\bar{U}_{K;\alpha})} \tag{4.24}$$

$$\eta \bar{M}^{\alpha 2K}_{(f)} = \rho \frac{\partial \bar{F}}{\partial (\psi_{(f)2K;\alpha})} \tag{4.25}$$

$$(1-\eta) M^{\alpha 2K}_{(m)} = \rho \frac{\partial \bar{F}}{\partial (\psi_{(m)2K;\alpha})} \tag{4.26}$$

$$\eta \bar{T}^{2K}_{(f)} = \rho \frac{\partial \bar{F}}{\partial (\psi_{(f)2K})} \tag{4.27}$$

$$(1-\eta) \bar{T}^{2K}_{(m)} = \rho \frac{\partial \bar{F}}{\partial (\psi_{(m)2K})} \tag{4.28}$$

where

$$\begin{aligned}U_{K;\alpha} &= \partial_\alpha U_K - \begin{Bmatrix} M \\ K\ \ \alpha \end{Bmatrix} U_M \\ \psi_{(f)2K;\alpha} &= \partial_\alpha \psi_{(f)2K} - \begin{Bmatrix} M \\ K\ \ \alpha \end{Bmatrix}_{\mathbf{G}} \psi_{(f)2M} \\ \psi_{(m)2K;\alpha} &= \partial_\alpha \psi_{(m)2K} - \begin{Bmatrix} M \\ K\ \ \alpha \end{Bmatrix}_{\mathbf{G}} \psi_{(f)2M}.\end{aligned} \tag{4.29}$$

## 5. LINEAR THEORY OF AN ISOTROPIC CURVILINEAR LAMINATED COMPOSITE

For most applications the infinitesimal theory of elasticity is quite adequate and considerably simpler than the finite theory. In this section we present the infinitesimal theory of an isotropic curvilinear composite. If the strains and rotations are small, then the strain energies of the $k$th reinforcing and matrix layers can be approximated by

$$\begin{aligned}\rho_f F_{(fk)} &= \frac{\lambda_f}{2} (\tilde{E}_{(fk)}{}^{K}{}_{K})^2 + \mu_f \tilde{E}_{(fk)}{}^{K}{}_{L} \tilde{E}_{(fk)}{}^{L}{}_{K} \\ \rho_m F_{(mk)} &= \frac{\lambda_m}{2} (\tilde{E}_{(mk)}{}^{K}{}_{K})^2 + \mu_m \tilde{E}_{(mk)}{}^{K}{}_{L} \tilde{E}_{(mk)}{}^{L}{}_{K},\end{aligned} \tag{5.1}$$

where $\tilde{E}_{(fk)KL}$ and $\tilde{E}_{(mk)KL}$ are the infinitesimal strain tensor of the $k$th reinforcing and matrix layers, respectively:

$$\begin{aligned}\tilde{E}_{(fk)KL} &= \tfrac{1}{2}(U^{(fk)}_{K;L}+U^{(fk)}_{L;K})\\ \tilde{E}_{(mk)KL} &= \tfrac{1}{2}(U^{(mk)}_{K;L}+U^{(mk)}_{L;K}).\end{aligned} \tag{5.2}$$

From (2.29) we see that

$$\begin{aligned}\tilde{E}_{(fk)\alpha\beta} &= \tfrac{1}{2}(\bar{U}^{(fk)}_{\alpha;\beta}+\bar{U}^{(fk)}_{\beta;\alpha})+\tfrac{1}{2}\bar{X}^{2}_{(f)}(\psi^{(fk)}_{\alpha;\beta}+\psi^{(fk)}_{\beta;\alpha})\\ \tilde{E}_{(fk)2\beta} &= E_{(fk)\beta 2} = \tfrac{1}{2}(\bar{U}^{(fk)}_{2;\beta}+\psi^{(fk)}_{(f)2\beta})+\tfrac{1}{2}\bar{X}^{2}_{(f)}\psi_{(f)22;\beta}\\ \tilde{E}_{(fk)22} &= \psi_{(fk)22},\end{aligned} \tag{5.3}$$

and similarly for $\tilde{E}_{(mk)KL}$. Substituting (5.1) and (5.3) into (4.4), and passing to the continuum model, (4.10) for an infinitesimal isotropic laminated composite becomes:

$$\begin{aligned}&\rho\bar{F}(\bar{U}_{K;\alpha},\psi_{(f)2K;\alpha},\psi_{(m)2K;\alpha},\psi_{(f)2K},\psi_{(m)2K})\\ &= \tfrac{1}{2}\lambda(\bar{E}^{\alpha}{}_{\alpha})^2+\mu\bar{E}^{\alpha\beta}\bar{E}_{\beta\alpha}+\lambda_f\eta\bar{E}^{\alpha}{}_{\alpha}\psi_{(f)22}+\lambda_m(1-\eta)\bar{E}^{\alpha}{}_{\alpha}\psi_{(m)22}\\ &+\frac{(\lambda_f+2\mu_f)\eta}{2}(\psi_{(f)22})^2+\frac{(\lambda_m+2\mu_m)(1-\eta)}{2}(\psi_{(m)22})^2+\tfrac{1}{2}\mu\bar{G}^{\alpha\beta}\bar{U}_{2;\alpha}\bar{U}_{2;\beta}\\ &+\mu_f\eta\bar{G}^{\alpha\beta}\bar{U}_{2;\alpha}\psi_{(f)2\beta}+\mu_m(1-\eta)\bar{G}^{\alpha\beta}\bar{U}_{2;\alpha}\psi_{(m)2\beta}+\frac{\mu_f\eta}{2}\psi_{(f)2\alpha}\psi_{(f)}{}^{\alpha}{}_{2}\\ &+\frac{\mu_m(1-\eta)}{2}\psi_{(m)2\alpha}\psi_{(m)2}{}^{\alpha}+\frac{\eta\lambda_f J_f}{2}(E_{(f)}{}^{\alpha}{}_{2\alpha})^2+\frac{(1-\eta)\lambda_m J_m}{2}(E_{(m)}{}^{\alpha}{}_{2\alpha})^2\\ &+\eta\mu_f J_f E_{(f)}{}^{\alpha}{}_{2\beta}E_{(f)}{}^{\beta}{}_{2\alpha}+(1-\eta)\mu_m J_m E_{(m)}{}^{\alpha}{}_{2\beta}E_{(m)}{}^{\beta}{}_{2\alpha}\\ &+\frac{\eta\mu_f J_f}{2}\bar{G}^{\alpha\beta}\psi_{(f)22;\alpha}\psi_{(f)22;\beta}+\frac{(1-\eta)\mu_m J_m}{2}\bar{G}^{\alpha\beta}\psi_{(m)22;\alpha}\psi_{(m)22;\beta},\end{aligned} \tag{5.4}$$

where we have defined

$$\bar{E}_{\alpha\beta} = \tfrac{1}{2}(\bar{U}_{\alpha;\beta}+\bar{U}_{\beta;\alpha}) \tag{5.5}$$

$$E_{(f)\alpha 2\beta} = \tfrac{1}{2}(\psi_{(f)2\alpha;\beta}+\psi_{(f)2\beta;\alpha}) \tag{5.6}$$

$$E_{(m)\alpha 2\beta} = \tfrac{1}{2}(\psi_{(m)2\alpha;\beta}+\psi_{(m)2\beta;\alpha}) \tag{5.7}$$

and

$$\begin{aligned}\lambda &= \lambda_f\eta+\lambda_m(1-\eta)\\ \mu &= \mu_f\eta+\mu_m(1-\eta).\end{aligned} \tag{5.8}$$

Using (5.4) in (4.24)–(4.28) we obtain the following constitutive equations for an isotropic elastic curvilinear laminated composite:

$$\bar{T}_{\alpha\beta} = [\lambda\bar{E}^{\gamma}{}_{\gamma}+\lambda_f\eta\psi_{(f)22}+\lambda_m(1-\eta)\psi_{(m)22}]\bar{G}_{\alpha\beta}+2\mu\bar{E}_{\alpha\beta} \tag{5.9}$$

$$\bar{T}_{\beta 2} = \mu\bar{U}_{2;\beta}+\mu_f\eta\psi_{(f)2\beta}+\mu_m(1-\eta)\psi_{(m)2\beta} \tag{5.10}$$

$$M^{(f)}_{\alpha 2\beta} = J_f[\lambda_f E^{(f)\gamma}{}_{2\gamma}\bar{G}_{\alpha\beta}+2\mu_f E^{(f)}{}_{\alpha 2\beta}] \tag{5.11}$$

$$M^{(f)}_{\alpha 22} = J_f \mu_f \psi_{(f)22;\alpha} \tag{5.12}$$

$$\bar{T}^{(f)}_{2\beta} = \mu_f(\bar{U}_{2;\beta} + \psi_{(f)2\beta}) \tag{5.13}$$

$$\bar{T}^{(f)}_{22} = \lambda_f \bar{E}^{\gamma}{}_{\gamma} + (\lambda_f + 2\mu_f)\psi_{(f)22} \tag{5.14}$$

$$M^{(m)}_{\alpha 2\beta} = J_m[\lambda_m E^{(m)\gamma}{}_{2\gamma} \bar{G}_{\alpha\beta} + 2\mu_m E^{(m)}_{\alpha 2\beta}] \tag{5.15}$$

$$M^{(m)}_{\alpha 22} = J_m \mu_m \psi_{(m)22;\alpha} \tag{5.16}$$

$$\bar{T}^{(m)}_{2\beta} = \mu_m(\bar{U}_{2;\beta} + \psi_{(m)2\beta}) \tag{5.17}$$

$$T^{(m)}_{22} = \lambda_m \bar{E}^{\gamma}{}_{\gamma} + (\lambda_m + 2\mu_m)\psi_{(m)22}. \tag{5.18}$$

## 6. CYLINDRICAL AND SPHERICAL LAMINATED BODIES

The balance laws and constitutive equations presented in the previous sections have been written in tensor notation. In the application of these equations to a laminated composite with a specific geometry it is necessary to express the constraint condition (4.29), the balance laws (4.26)–(4.28) and the constitutive equations (5.9)–(5.18) in terms of the physical components of the tensors. In this section we carry this out for cylindrical and spherical laminated composites.

(a) *Cylindrical laminated bodies*†

We choose $X^1 = z$, $X^2 = r$ and $X^3 = \theta$. Thus the metric tensor and its inverse are given by

$$[\bar{G}_{KL}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{bmatrix} \qquad [\bar{G}^{KL}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{r^2} \end{bmatrix}. \tag{6.1}$$

and the nonvanishing Christoffel symbols are

$$\begin{Bmatrix} 3 \\ 2\ 3 \end{Bmatrix}_{\bar{G}} = \begin{Bmatrix} 3 \\ 3\ 2 \end{Bmatrix}_{\bar{G}} = \frac{1}{r}, \qquad \begin{Bmatrix} 2 \\ 3\ 3 \end{Bmatrix}_{\bar{G}} = -r. \tag{6.2}$$

We introduce the physical components $\bar{\mathbf{U}}$, $\boldsymbol{\psi}_{2(f)}$, $\boldsymbol{\psi}_{2(m)}$, $\bar{\mathbf{T}}^{\alpha}$, $\boldsymbol{\Sigma}^2$, $\mathbf{M}^{\alpha 2}_{(f)}$, $\mathbf{M}^{\alpha 2}_{(m)}$, $\bar{\mathbf{T}}^2_{(f)}$, $\bar{\mathbf{T}}^2_{(m)}$,

$$(\bar{U}^1, \bar{U}^2, \bar{U}^3) = \left(\bar{U}_z, \bar{U}_r, \frac{\bar{U}_\theta}{r}\right) \tag{6.3}$$

$$\begin{pmatrix} \bar{T}^{11} & \bar{T}^{12} & \bar{T}^{13} \\ \bar{T}^{31} & \bar{T}^{32} & \bar{T}^{33} \end{pmatrix} = \begin{pmatrix} \bar{T}_{zz} & \bar{T}_{zr} & \dfrac{\bar{T}_{z\theta}}{r} \\ \dfrac{\bar{T}_{\theta z}}{r} & \dfrac{\bar{T}_{\theta r}}{r} & \dfrac{\bar{T}_{\theta\theta}}{r^2} \end{pmatrix} \tag{6.4}$$

† A direct calculation of these equations has been given by Chou and Achenbach[8].

$$(\Sigma^{21}, \Sigma^{22}, \Sigma^{23}) = \left(\Sigma_{rz}, \Sigma_{rr}, \frac{\Sigma_{r\theta}}{r}\right) \tag{6.5}$$

$$(\psi_{(f)2}{}^{1}, \psi_{(f)2}{}^{2}, \psi_{(f)2}{}^{3}) = \left(\psi_{2z}^{(f)}, \psi_{2r}^{(f)}, \frac{\psi_{2\theta}^{(f)}}{r}\right) \tag{6.6}$$

$$\begin{pmatrix} M_{(f)}^{121} & M_{(f)}^{22} & M_{(f)}^{123} \\ \\ M_{(f)}^{321} & M_{(f)}^{322} & M_{(f)}^{323} \end{pmatrix} = \begin{pmatrix} M_{z2z}^{(f)}, M_{z2r}^{(f)}, \dfrac{M_{z2\theta}^{(f)}}{r} \\ \dfrac{M_{\theta 2z}^{(f)}}{r}, \dfrac{M_{\theta 2r}^{(f)}}{r}, \dfrac{M_{\theta 2\theta}^{(f)}}{r^2} \end{pmatrix} \tag{6.7}$$

$$(\overline{T}_{(f)}^{21}, \overline{T}_{(f)}^{22}, \overline{T}_{(f)}^{23}) = \left(\overline{T}_{2z}^{(f)}, \overline{T}_{2r}^{(f)}, \frac{\overline{T}_{2\theta}^{(f)}}{r}\right), \tag{6.8}$$

etc.

We can write (4.29), (4.26)–(4.28) and (5.9)–(5.18) as

*Constraint conditions.*

$$\begin{aligned} \partial_r \overline{U}_r &= \eta\psi_{2r}^{(f)} + (1-\eta)\psi_{2r}^{(m)} \\ \partial_r \overline{U}_\theta &= \eta\psi_{2\theta}^{(f)} + (1-\eta)\psi_{2\theta}^{(m)} \\ \partial_r \overline{U}_z &= \eta\psi_{2z}^{(f)} + (1-\eta)\psi_{2z}^{(m)}. \end{aligned} \tag{6.9}$$

*Balance of linear momentum.*

$$\begin{aligned} \partial_r \Sigma_{rr} + \frac{1}{r}\Sigma_{rr} + \frac{1}{r}\partial_\theta \overline{T}_{\theta r} + \partial_z \overline{T}_{zr} - \frac{1}{r}\overline{T}_{\theta\theta} + \rho\bar{f}_r &= \rho\ddot{\overline{U}}_r \\ \partial_r \Sigma_{r\theta} + \frac{1}{r}\Sigma_{r\theta} + \frac{1}{r}\partial_\theta \overline{T}_{\theta\theta} + \partial_z \overline{T}_{z\theta} + \frac{1}{r}\overline{T}_{\theta r} + \rho\bar{f}_\theta &= \rho\ddot{\overline{U}}_\theta \\ \partial_z \Sigma_{rz} + \frac{1}{r}\Sigma_{rz} + \frac{1}{r}\partial_\theta \overline{T}_{\theta z} + \partial_z \overline{T}_{zz} + \rho\bar{f}_z &= \rho\ddot{\overline{U}}_z. \end{aligned} \tag{6.10}$$

*Moment of momentum.*

$$\begin{aligned} \frac{1}{r}\partial_\theta M_{\theta 2r}^{(f)} + \partial_z M_{z2r}^{(f)} - \frac{1}{r}M_{\theta 2\theta}^{(f)} + \Sigma_{rr} - \overline{T}_{2r}^{(f)} + \rho_f l_{2r}^{(f)} &= \rho_f J_f \ddot{\psi}_{2r}^{(f)} \\ \frac{1}{r}\partial_\theta M_{\theta 2\theta}^{(f)} + \partial_z M_{z2\theta}^{(f)} + \frac{1}{r}M_{\theta 2r}^{(f)} + \Sigma_{r\theta} - \overline{T}_{2\theta}^{(f)} + \rho_f l_{2\theta}^{(f)} &= \rho_f J_f \ddot{\psi}_{2\theta}^{(f)} \\ \frac{1}{r}\partial_\theta M_{\theta 2z}^{(f)} + \partial_z M_{z2z}^{(f)} + \Sigma_{rz} - \overline{T}_{2z}^{(f)} + \rho_f l_{2z}^{(f)} &= \rho_f J_f \ddot{\psi}_{2z}^{(f)} \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} \frac{1}{r}\partial_\theta M_{\theta 2r}^{(m)} + \partial_z M_{z2r}^{(m)} - \frac{1}{r}M_{\theta 2\theta}^{(m)} + \Sigma_{rr} - \overline{T}_{2r}^{(m)} + \rho_m l_{2r}^{(m)} &= \rho_m J_m \ddot{\psi}_{2r}^{(m)} \\ \frac{1}{r}\partial_\theta M_{\theta 2\theta}^{(m)} + \partial_z M_{z2\theta}^{(m)} + \frac{1}{r}M_{\theta 2r}^{(m)} + \Sigma_{r\theta} - \overline{T}_{2\theta}^{(m)} + \rho_m l_{2\theta}^{(m)} &= \rho_m J_m \ddot{\psi}_{2\theta}^{(m)} \\ \frac{1}{r}\partial_\theta M_{\theta 2z}^{(m)} + \partial_z M_{z2z}^{(m)} + \Sigma_{rz} - \overline{T}_{2z}^{(m)} + \rho_m l_{2z}^{(m)} &= \rho_m J_m \ddot{\psi}_{2z}^{(m)}. \end{aligned} \tag{6.12}$$

*Constitutive equations.*

Gross stress tensor

$$\bar{T}_{\theta\theta} = \lambda\partial_z\bar{U}_z + \lambda_f\eta\psi^{(f)}_{2r} + \lambda_m(1-\eta)\psi^{(m)}_{2r} + (\lambda+2\mu)\left(\frac{1}{r}\partial_\theta\bar{U}_\theta + \frac{1}{r}\bar{U}_r\right)$$

$$\bar{T}_{zz} = \lambda\left(\frac{1}{r}\partial_\theta\bar{U}_\theta + \frac{1}{r}\bar{U}_r\right) + \lambda_f\eta\psi^{(f)}_{2r} + \lambda_m(1-\eta)\psi^{(m)}_{2r} + (\lambda+2\mu)\partial_z\bar{U}_z$$

$$\bar{T}_{\theta z} = \bar{T}_{z\theta} = \mu\left(\frac{1}{r}\partial_\theta\bar{U}_z + \partial_z\bar{U}_\theta\right) \tag{6.13}$$

$$\bar{T}_{\theta r} = \mu\left(\frac{1}{r}\partial_\theta\bar{U}_r - \frac{\bar{U}_\theta}{r}\right) + \mu_f\eta\psi^{(f)}_{2\theta} + \mu_m(1-\eta)\psi^{(m)}_{2\theta}$$

$$\bar{T}_{zr} = \mu\partial_z\bar{U}_r + \mu_f\eta\psi^{(f)}_{2r} + \mu_m(1-\eta)\psi^{(m)}_{2r}.$$

Moment of stress

$$M^{(f)}_{\theta 2\theta} = J_f\left[\lambda_f\partial_z\psi^{(f)}_{2z} + (\lambda_f+2\mu_f)\left(\frac{1}{r}\partial_\theta\psi^{(f)}_{2\theta} + \frac{1}{r}\psi^{(f)}_{2r}\right)\right]$$

$$M^{(f)}_{z2z} = J_f\left[\lambda_f\left(\frac{1}{r}\partial_\theta\psi^{(f)}_{2\theta} + \frac{1}{r}\psi^{(f)}_{2r}\right) + (\lambda_f+2\mu_f)\partial_z\psi^{(f)}_{2z}\right]$$

$$M^{(f)}_{\theta 2z} = M^{(f)}_{z2\theta} = J_f\mu_f\left(\frac{1}{r}\partial_\theta\psi^{(f)}_{2z} + \partial_z\psi^{(f)}_{2\theta}\right) \tag{6.14}$$

$$M^{(f)}_{\theta 2r} = J_f\mu_f\left(\frac{1}{r}\partial_\theta\psi^{(f)}_{2r} - \frac{\psi^{(f)}_{2\theta}}{r}\right)$$

$$M^{(f)}_{z2r} = J_f\mu_f\partial_z\psi^{(f)}_{2r}$$

Stress average

$$\bar{T}^{(f)}_{2\theta} = \mu_f\left(\frac{1}{r}\partial_\theta\bar{U}_r - \frac{\bar{U}_\theta}{r} + \psi^{(f)}_{2\theta}\right)$$

$$\bar{T}^{(f)}_{2z} = \mu_f(\partial_z\bar{U}_r + \psi^{(f)}_{2z}) \tag{6.15}$$

$$\bar{T}^{(f)}_{2r} = \lambda_f\left(\frac{1}{r}\partial_\theta\bar{U}_\theta + \frac{1}{r}\bar{U}_r + \partial_z\bar{U}_z\right) + (\lambda_f+2\mu_f)\psi^{(f)}_{2r}.$$

(b) *Spherical laminated bodies*

We choose $X^1 = \varphi$, $X^2 = r$, $X^3 = \theta$, where $r$ is the length of the radius vector, $\theta$ is the angle between the radius vector and the $z$-axis and $\varphi$ is the angle between the projection in the $X$–$Y$ plane of the radius vector and the $X$-axis. Thus the metric tensor and its inverse are given by

$$[\bar{G}_{KL}] = \begin{bmatrix} r^2\sin^2\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{bmatrix} \qquad [\bar{G}^{KL}] = \begin{bmatrix} \dfrac{1}{r^2\sin^2\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{1}{r^2} \end{bmatrix} \tag{6.16}$$

and the nonvanishing Christoffel symbols are

$$\begin{aligned}
&\left\{\begin{matrix} 3 \\ 1 \quad 1 \end{matrix}\right\}_{\bar{G}} = -\sin\theta\cos\theta, &&\left\{\begin{matrix} 1 \\ 2 \quad 1 \end{matrix}\right\}_{\bar{G}} = \left\{\begin{matrix} 1 \\ 1 \quad 2 \end{matrix}\right\}_{\bar{G}} = \frac{1}{r}, \\
&\left\{\begin{matrix} 1 \\ 3 \quad 1 \end{matrix}\right\}_{\bar{G}} = \left\{\begin{matrix} 1 \\ 1 \quad 3 \end{matrix}\right\}_{\bar{G}} = \cot\theta, &&\left\{\begin{matrix} 2 \\ 3 \quad 3 \end{matrix}\right\}_{\bar{G}} = -r, \\
&\left\{\begin{matrix} 2 \\ 1 \quad 1 \end{matrix}\right\}_{\bar{G}} = -r\sin^2\theta, &&\left\{\begin{matrix} 3 \\ 3 \quad 2 \end{matrix}\right\}_{\bar{G}} = \left\{\begin{matrix} 3 \\ 2 \quad 3 \end{matrix}\right\}_{\bar{G}} = \frac{1}{r}.
\end{aligned} \tag{6.17}$$

We introduce the physical components of $\bar{\mathbf{U}}$, $\boldsymbol{\psi}_{(f)2}$, etc., as

$$(\bar{U}^1, \bar{U}^2, \bar{U}^3) = \left(\frac{\bar{U}_\varphi}{r\sin\theta}, \bar{U}_r, \frac{\bar{U}_\theta}{r}\right) \tag{6.18}$$

$$\begin{pmatrix} \bar{T}^{11} & \bar{T}^{12} & \bar{T}^{13} \\ \bar{T}^{31} & \bar{T}^{32} & \bar{T}^{33} \end{pmatrix} = \begin{pmatrix} \dfrac{\bar{T}_{\varphi\varphi}}{r^2\sin^2\theta} & \dfrac{\bar{T}_{\varphi r}}{r\sin\theta} & \dfrac{\bar{T}_{\varphi\theta}}{r^2\sin\theta} \\ \dfrac{\bar{T}_{\theta\varphi}}{r^2\sin\theta} & \dfrac{\bar{T}_{\theta r}}{r} & \dfrac{\bar{T}_{\theta\theta}}{r^2} \end{pmatrix} \tag{6.19}$$

$$(\Sigma^{21}, \Sigma^{22}, \Sigma^{23}) = \left(\frac{\Sigma_{r\varphi}}{r\sin\theta}, \Sigma_{rr}, \Sigma_{r\theta}\right) \tag{6.20}$$

$$(\psi_{(f)2}{}^1, \psi_{(f)2}{}^2, \psi_{(f)2}{}^3) = \left(\frac{\psi^{(f)}_{2\varphi}}{r\sin\theta}, \psi^{(f)}_{2r}, \frac{\psi^{(f)}_{2\theta}}{r}\right) \tag{6.21}$$

$$\begin{pmatrix} M^{121}_{(f)} & M^{122}_{(f)} & M^{123}_{(f)} \\ M^{321}_{(f)} & M^{322}_{(f)} & M^{323}_{(f)} \end{pmatrix} = \begin{pmatrix} \dfrac{M^{(f)}_{\varphi 2\varphi}}{r^2\sin^2\theta} & \dfrac{M^{(f)}_{\varphi 2r}}{r\sin\theta} & \dfrac{M^{(f)}_{\varphi 2\theta}}{r^2\sin\theta} \\ \dfrac{M^{(f)}_{\theta 2\varphi}}{r^2\sin\theta} & \dfrac{M^{(f)}_{\theta 2r}}{r} & \dfrac{M^{(f)}_{\theta 2\theta}}{r^2} \end{pmatrix} \tag{6.22}$$

$$(\bar{T}^{21}_{(f)}, \bar{T}^{22}_{(f)}, \bar{T}^{23}_{(f)}) = \left(\frac{\bar{T}^{(f)}_{2\varphi}}{r\sin\theta}, \bar{T}^{(f)}_{2r}, \frac{\bar{T}^{(f)}_{2\theta}}{r}\right), \tag{6.23}$$

etc.

Using (6.16)–(6.23) in (4.29), (4.26)–(4.28) and (5.9)–(5.18), we have

*Constraint conditions.*

$$\begin{aligned}
\partial_r\bar{U}_r &= \eta\psi^{(f)}_{2r} + (1-\eta)\psi^{(m)}_{2r} \\
\partial_r\bar{U}_\theta &= \eta\psi^{(f)}_{2\theta} + (1-\eta)\psi^{(m)}_{2\theta} \\
\partial_r\bar{U}_\varphi &= \eta\psi^{(f)}_{2\varphi} + (1-\eta)\psi^{(m)}_{2\varphi}.
\end{aligned} \tag{6.24}$$

*Balance of linear momentum.*

$$\begin{aligned}
&\partial_r\Sigma_{rr} + \frac{2}{r}\Sigma_{rr} + \frac{1}{r}\partial_\theta\bar{T}_{\theta r} + \frac{1}{r\sin\theta}\partial_\varphi\bar{T}_{\varphi r} + \frac{1}{r}(\cot\theta\bar{T}_{\theta r} - \bar{T}_{\theta\theta} - \bar{T}_{\varphi\varphi}) + \rho\bar{f}_r = \rho\ddot{\bar{U}}_r \\
&\partial\Sigma_{r\theta} + \frac{2}{r}\Sigma_{r\theta} + \frac{1}{r}\partial_\theta\bar{T}_{\theta\theta} + \frac{1}{r\sin\theta}\partial_\varphi\bar{T}_{\varphi\theta} + \frac{1}{r}(\bar{T}_{\theta r} + \cot\theta\bar{T}_{\theta\theta} - \cot\theta\bar{T}_{\varphi\varphi}) + \rho\bar{f}_\theta = \rho\ddot{\bar{U}}_\theta \\
&\partial\Sigma_{r\varphi} + \frac{2}{r}\Sigma_{r\varphi} + \frac{1}{r}\partial_\theta\bar{T}_{\theta\varphi} + \frac{1}{r\sin\theta}\partial_\varphi\bar{T}_{\varphi\varphi} + \frac{1}{r}(\bar{T}_{\varphi r} + \cot\theta\bar{T}_{\varphi\theta} + \cot\theta\bar{T}_{\theta\varphi}) + \rho\bar{f}_\varphi = \rho\ddot{\bar{U}}_\varphi.
\end{aligned} \tag{6.25}$$

*Moment of momentum.*

$$\frac{1}{r}\partial_\theta M^{(f)}_{\theta 2r}+\frac{1}{r\sin\theta}\partial_\varphi M^{(f)}_{\varphi 2r}+\frac{1}{r}(\cot\theta M^{(f)}_{\theta 2r}-M^{(f)}_{\theta 2\theta}-M^{(f)}_{\varphi 2\varphi})+\Sigma_{rr}-\overline{T}^{(f)}_{2r}+\rho_f l^{(f)}_{2r} = \rho_f J_f \ddot{\psi}^{(f)}_{2r}$$

$$\frac{1}{r}\partial_\theta M^{(f)}_{\theta 2\theta}+\frac{1}{r\sin\theta}\partial_\varphi M^{(f)}_{\varphi 2\theta}+\frac{1}{r}(M^{(f)}_{\theta 2r}+\cot\theta M^{(f)}_{\theta 2\theta}-\cot\theta M^{(f)}_{\varphi 2\varphi})+\Sigma_{r\theta}-\overline{T}^{(f)}_{2\theta}+\rho_f l^{(f)}_{2\theta}$$

$$= \rho_f J_f \ddot{\psi}^{(f)}_{2\theta} \qquad (6.26)$$

$$\frac{1}{r}\partial_\theta M^{(f)}_{\theta 2\varphi}+\frac{1}{r\sin\theta}\partial_\varphi M^{(f)}_{\varphi 2\varphi}-\frac{1}{r}(M^{(f)}_{\varphi 2r}+\cot\theta M^{(f)}_{\varphi 2\theta}+\cot\theta M^{(f)}_{\theta 2\varphi})+\Sigma_{r\varphi}-\overline{T}^{(f)}_{2\varphi}+\rho_f l^{(f)}_{2\varphi}$$

$$= \rho_f J_f \ddot{\psi}^{(f)}_{2\varphi}$$

and

$$\frac{1}{r}\partial_\theta M^{(m)}_{\theta 2r}+\frac{1}{r\sin\theta}\partial_\varphi M^{(m)}_{\varphi 2r}+\frac{1}{r}(\cot\theta M^{(m)}_{\theta 2r}-M^{(m)}_{\theta 2\theta}-M^{(m)}_{\varphi 2\varphi})+\Sigma_{rr}-\overline{T}^{(m)}_{2r}+\rho_m l^{(m)}_{2r} = \rho_m J_m \ddot{\psi}^{(m)}_{2r}$$

$$\frac{1}{r}\partial_\theta M^{(m)}_{\theta 2\theta}+\frac{1}{r\sin\theta}\partial_\varphi M^{(m)}_{\varphi 2\theta}+\frac{1}{r}(M^{(m)}_{\theta 2r}+\cot\theta M^{(m)}_{\theta 2\theta}-\cot\theta M^{(m)}_{\varphi 2\varphi})+\Sigma_{r\theta}-\overline{T}^{(m)}_{2\theta}+\rho_m l^{(m)}_{2\theta}$$

$$= \rho_m J_m \ddot{\psi}^{(m)}_{2\theta} \qquad (6.27)$$

$$\frac{1}{r}\partial_\theta M^{(m)}_{\theta 2\varphi}+\frac{1}{r\sin\theta}\partial_\varphi M^{(m)}_{\varphi 2\varphi}+\frac{1}{r}(M^{(m)}_{\varphi 2r}+\cot\theta M^{(m)}_{\varphi 2\theta}+\cot\theta M^{(m)}_{\theta 2\varphi})+\Sigma_{r\varphi}-\overline{T}^{(m)}_{2\varphi}+\rho_m l^{(m)}_{2\varphi}$$

$$= \rho_m J_m \ddot{\psi}^{(m)}_{2\varphi}.$$

*Constitutive relations.*

Gross stress tensor

$$\overline{T}_{\theta\theta} = \lambda\left(\frac{1}{r\sin\theta}\partial_\varphi \overline{U}_\varphi+\frac{\overline{U}_r}{r}+\overline{U}_\theta\frac{\cot\theta}{r}\right)+\lambda_f\eta\psi^{(f)}_{2r}+\lambda_m(1-\eta)\psi^{(m)}_{2r}+(\lambda+2\mu)\left(\frac{1}{r}\partial_\theta\overline{U}_\theta+\frac{\overline{U}_r}{r}\right)$$

$$\overline{T}_{\varphi\varphi} = \lambda\left(\frac{1}{r}\partial_\theta\overline{U}_\theta+\frac{\overline{U}_r}{r}\right)+\lambda_f\eta\psi^{(f)}_{2r}+\lambda_m(1-\eta)\psi^{(m)}_{2r}+(\lambda+2\mu)\left(\frac{1}{r\sin\theta}\partial_\varphi\overline{U}_\varphi+\frac{\overline{U}_r}{r}+\overline{U}_\theta\frac{\cot\theta}{r}\right)$$

$$\overline{T}_{\theta\varphi} = \overline{T}_{\varphi\theta} = \mu\left(\frac{1}{r}\partial_\theta\overline{U}_\varphi-\frac{\overline{U}_\varphi\cot\theta}{r}+\frac{1}{r\sin\theta}\partial_\varphi\overline{U}_\theta\right)$$

$$\overline{T}_{\theta r} = \mu\left(\frac{1}{r}\partial_\theta\overline{U}_r-\frac{\overline{U}_\theta}{r}\right)+\mu_f\eta\psi^{(f)}_{2\theta}+\mu_m(1-\eta)\psi^{(m)}_{2\theta} \qquad (6.28)$$

$$\overline{T}_{\varphi r} = \mu\left(\frac{1}{r\sin\theta}\partial_\varphi\overline{U}_r-\frac{\overline{U}_\varphi}{r}\right)+\mu_f\eta\psi^{(f)}_{2\varphi}+\mu_m(1-\eta)\psi^{(m)}_{2\varphi}.$$

Moment of stress

$$M^{(f)}_{\theta 2\theta} = J_f\left\{\lambda_f\left(\frac{1}{r\sin\theta}\partial_\varphi\psi^{(f)}_{2\varphi}+\frac{\psi^{(f)}_{2r}}{r}+\psi^{(f)}_{2\theta}\frac{\cot\theta}{r}\right)+(\lambda_f+2\mu_f)\left(\frac{1}{r}\partial_\theta\psi^{(f)}_{2\theta}+\frac{\psi^{(f)}_{2r}}{r}\right)\right\}$$

$$M^{(f)}_{\varphi 2\varphi} = J_f\left\{\lambda_f\left(\frac{1}{r}\partial_\theta\psi^{(f)}_{2\theta}+\frac{\psi^{(f)}_{2r}}{r}\right)+(\lambda_f+2\mu_f)\left(\frac{1}{r\sin\theta}\partial_\varphi\psi^{(f)}_{2\varphi}+\frac{\psi^{(f)}_{2r}}{r}+\psi^{(f)}_{2\theta}\frac{\cot\theta}{r}\right)\right\}$$

$$M^{(f)}_{\theta 2\varphi} = M^{(f)}_{\varphi\theta} = J_f\mu_f\left(\frac{1}{r}\partial_\theta\psi^{(f)}_{2\varphi}-\frac{\psi^{(f)}_{2\varphi}\cot\theta}{r}+\frac{1}{r\sin\theta}\partial_\varphi\psi^{(f)}_{2\theta}\right) \tag{6.29}$$

$$M^{(f)}_{\theta 2r} = J_f\mu_f\left(\frac{1}{r}\partial_\theta\psi^{(f)}_{2r}-\frac{\psi^{(f)}_{2\theta}}{r}\right)$$

$$M^{(f)}_{\varphi 2r} = J_f\mu_f\left(\frac{1}{r\sin\theta}\partial_\varphi\psi^{(f)}_{2r}-\frac{\psi^{(f)}_{2\varphi}}{r}\right).$$

Stress average

$$\bar{T}^{(f)}_{2\theta} = \mu_f\left(\frac{1}{r}\partial_\theta\bar{U}_r-\frac{U_\theta}{r}+\psi^{(f)}_{2\theta}\right)$$

$$\bar{T}^{(f)}_{2\varphi} = \mu_f\left(\frac{1}{r\sin\theta}\partial_\varphi\bar{U}_r-\frac{\bar{U}_\varphi}{r}+\psi^{(f)}_{2\varphi}\right) \tag{6.30}$$

$$\bar{T}^{(f)}_{2r} = \lambda_f\left(\frac{1}{r}\partial_\theta\bar{U}_\theta+\frac{1}{r\sin\theta}\partial_\varphi\bar{U}_\varphi+\frac{\partial\bar{U}_r}{r}+\bar{U}_\theta\cot\theta\right)+(\lambda_f+2\mu_f)\psi^{(f)}_{2r}.$$

and similarly for the matrix.

## 7. CONCLUDING REMARKS

We have presented, in the form of a continuum model, the kinematics, dynamics and constitutive equations of a curvilinear laminated material. The kinematics and dynamical balance laws have been derived in such a way that they are independent of the material composition of the layers. Though we have presented the constitutive theory for elastic materials, the treatment for more general classes of materials follows readily. In particular, for linear viscoelastic materials one need only replace the constants in equations (5.9)–(5.18) by their corresponding convolution integrals (cf. [3]). The inclusion of thermodynamical effects requires an analysis of the temperature and the energy equation similar to the one presented in [3]. We leave these generalizations to future work.

One can, of course, question the validity of a two term expansion in describing the behavior of a curvilinear laminated body. Ultimately the proof of such an approach rests with a comparison of the results of the approximate theory presented here with either exact solutions and/or experimental investigations. However, it is expected that if the ratio of the thickness of the layering to the radius of curvature of the layering is small, the approximated theory presented here should be as adequate as the approximate theories of plane laminates presented in Refs. [1–4].

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**Абстракт**—Даются кинематические, динамические и конститутивные уравнения приближенной теории для криволинейного сложного материала. Начиная двумя членами разложения переменных поля вокруг серединных поверхностей дискретных слоев, показано, что деформация криволинейного слоистого материала в первом приближении описана полями трех векторов, учитывающими полное движение и локальные деформации. Определяются закон динамического равновесия из суммарных напряжений и моментов напряжений. Даются формулы конститутивной теории для нелинейных упругих материалов. Обсуждается упрощенный вариант линейных конститутивных уравнений. Даются специфические формы законов равновесия и конститутивных уравнений для цилиндрических и сферических слоистых тел.